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A martingale decomposition of discrete Markov chains

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HIGHLIGHTS

- We consider a multivariate time series given from a discrete Markov chain.
- Its martingale decomposition is derived, with all terms given in closed form.
- The decomposition is analogous to the Beveridge–Nelson decomposition.
- Decomposition has three terms: a persistent, a transitory, and a deterministic trend.
- The autocovariance structure across all terms is fully characterized.

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ABSTRACT

We consider a multivariate time series whose increments are given from a homogeneous Markov chain. We show that the martingale component of this process can be extracted by a filtering method and establish the corresponding martingale decomposition in closed-form. This representation is useful for the analysis of time series that are confined to a grid, such as financial high frequency data.

1. Introduction

We consider a d-dimensional time series, $\{X_t\}$, whose increments, $\Delta X_t = X_t - X_{t-1}$, follow a homogeneous ergodic Markov chain with a countable state space. Thus, $X_t = X_0 + \sum_{j=1}^{t} \Delta X_j$, which makes X_t a (possibly non-stationary) Markov chain on a countable state space. We consider, $\mathrm{E}(X_{t+h}|\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$, is the natural filtration. The limit, as $h \to \infty$, is particularly interesting, because it leads to a martingale decomposition,

$$X_t = Y_t + \mu_t + U_t,$$

where μ_t is a linear deterministic trend, $\{Y_t, \mathcal{F}_t\}$ is a martingale with $Y_t = \lim_{h\to\infty} \mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t)$, and U_t is a bounded stationary process. We derive closed-form expressions for all terms in the representation of X_t.

The martingale decomposition of finite Markov chains is akin to the Beveridge–Nelson decomposition for ARIMA processes, see Beveridge and Nelson (1981),[1] and the Granger representation for vector autoregressive processes, see Johansen (1991). The decomposition has many applications, as the long-run properties of X_t are governed by the persistent component, Y_t, while U_t characterizes the transitory component of X_t. In macro-econometrics Y_t and U_t are often called "trend" and "cycle", respectively, with Y_t being interpreted as the long run growth while U_t defines the fluctuations around the growth path, see, e.g. Low and Anderson (2008). A martingale decomposition of a stochastic discount process can be used to disentangle economic components with long term and short run impact on asset valuation, see Hansen (2012). For the broader

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[1] The result, known as Beveridge–Nelson decomposition, appeared earlier in the statistics literature, e.g. Fuller (1976, Theorem 8.5.1). See Phillips and Solo (1992) for further discussion. The martingale decomposition is also key for the central limit theorem for stationary processes by Gordin (1969).

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concept of signal extraction of the "trend", see Harvey and Koopman (2002).

In the context with high-frequency financial data (which often are confined to a grid), Y_t and U_t may be labelled the efficient price and market microstructure noise, respectively. One could use the decomposition to estimate the quadratic variation of the latent efficient price Y_t, as in Large (2011) and Hansen and Horel (2009), and the framework could be adapted to study market information share, see e.g. Hasbrouck (1995). Markov processes are often used to approximate autoregressive processes in dynamic optimization problems, see Tauchen (1986) and Adda and Cooper (2000), and the decomposition could be used to compare the long-run properties of the approximating Markov process with those of the autoregressive process.

The paper is organized as follows: We establish an expression for the filtered process within the Markov chain framework, in Section 2, which leads to the martingale decomposition. Concluding remarks with discussion of various extensions are given in Section 3, and all proofs are given in the Appendix.

2. Theoretical framework

In this section we show how the observed process, $X_0, X_1, \ldots, X_n$, can be filtered in a Markov chain framework, using the natural filtration $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$. This leads to a martingale decomposition for X_t that is useful for a number of things.

Initially we seek the filtered price, $\mathrm{E}(X_{t+h}|\mathcal{F}_t)$, and we use the limit, as $h \to \infty$, to define the process,

$$Y_t = \lim_{h\to\infty} \mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t),$$

where $\mu_t = t\mu$ with $\mu = \mathrm{E}(\Delta X_t)$. We will show that $\{Y_t, \mathcal{F}_t\}$ is a martingale, in fact, Y_t is the martingale component of X_t that, in turn, reveals a martingale representation theorem for finite Markov processes.

Note that the one step increments of $\mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t)$ are, in general, autocorrelated at all order (including those lower than h), however all autocorrelations vanish as $h \to \infty$ and the martingale property of Y emerges. This filtering argument can be applied to any I(1) process for which $\mathrm{E}(\Delta X_{t+h}|\mathcal{F}_t) \overset{\text{a.s.}}{\to} \mathrm{E}(\Delta X_t)$ as $h \to \infty$, and this is the basic principle that Beveridge and Nelson (1981) used to extract the (stochastic) trend component of ARIMA processes.

2.1. Notation and assumptions

In this section we review the Markov terminology and present our notation that largely follows that in Brémaud (1999, Chapter 6). The following assumption is the only assumption we need to make.

Assumption 1. The increments $\{\Delta X_t\}_{t=1}^n$ are ergodic and distributed as a homogeneous Markov chain of order $k < \infty$, with $S < \infty$ states.

The assumption that S is finite can be dispensed with, which we detail in Section 3. For now we will assume S to be finite because it greatly simplifies the exposition. The transition matrix for price increments is denoted by P. For a Markov chain of order k with S basic states, P will be an $S^k \times S^k$ matrix. We use $\pi \in \mathbb{R}^{S^k}$ to denote the stationary distribution associated with P, which is uniquely defined by $\pi'P = \pi'$. The *fundamental matrix* is defined by[2]

$$Z = (I - P + \Pi)^{-1},$$

where $\Pi = \iota\pi'$ is a square matrix and $\iota = (1, \ldots, 1)'$, (so all rows of Π are simply π'). We use e_r to denote the r-th unit vector, so that $e_r'A$ is the r-th row of a matrix A of proper dimensions.

Let $\{x_1, \ldots, x_S\}$ be the support for ΔX_t, with $x_s \in \mathbb{R}^d$. We will index the possible realizations for the k-tuple, $\Delta\mathcal{X}_t = (\Delta X_{t-k+1}, \ldots, \Delta X_t)$, by $\mathbf{x}_s$, $s = 1, \ldots, S^k$, which includes all the perturbations, $(x_{i_1}, \ldots, x_{i_k})$, $i_1, \ldots, i_k = 1, \ldots, S$. The transition matrix, P, is given by

$$P_{r,s} = \Pr(\Delta\mathcal{X}_{t+1} = \mathbf{x}_s|\Delta\mathcal{X}_t = \mathbf{x}_r).$$

This matrix will be sparse when $k > 1$, because at most S transitions from any state have non-zero probability, regardless of the order of the Markov chain.

For notational reasons it is convenient to introduce the sequence $\{s_t\}$ that is defined by $\Delta\mathcal{X}_t = \mathbf{x}_{s_t}$, so that s_t denotes the observed state at time t. We also define the matrix $f \in \mathbb{R}^{S^k\times d}$ whose s-th row, denoted $f_s = e_s'f$, is the realization of $\Delta X'$ in state s. It follows that $\Delta X_t = f'e_{s_t}$ and that the expected value of the increments is given by $\mu = \mathrm{E}(\Delta X_t) = f'\pi \in \mathbb{R}^d$.

The auxiliary vector process, e_{s_t}, is such that $\mathrm{E}(e_{s_{t+1}}|\mathcal{F}_t) = P'e_{s_t}$, so that e_{s_t} can be expressed as a vector autoregressive process of order one with martingale difference innovations, see e.g. Hamilton (1994, p. 679).

2.2. Markov chain Filtering

The filtered process $\mathrm{E}(X_{t+h}|\mathcal{F}_t)$, is simple to compute in the Markov setting, because $\mathrm{E}(X_{t+h}|\mathcal{F}_t) = \mathrm{E}(X_{t+h}|\Delta\mathcal{X}_t)$ and $X_{t+h} = X_t + \sum_{j=1}^h \Delta X_{t+j}$ with $\mathrm{E}(\Delta X_{t+1}'|\Delta\mathcal{X}_t = \mathbf{x}_r) = \sum_{s=1}^{S^k} P_{r,s}f_s = e_r'Pf$. More generally we have $\mathrm{E}(\Delta X_{t+h}'|\Delta\mathcal{X}_t) = e_{s_t}'P^hf$, which shows that

$$\mathrm{E}(X_{t+h}'|\Delta\mathcal{X}_t) = X_t' + e_{s_t}'\sum_{j=1}^h P^jf.$$

After subtracting the deterministic trend, μ_{t+h}, we let $h \to \infty$ and define

$$Y_t = \lim_{h\to\infty} \mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t),$$

which we label the filtered process of X_t. The process, Y_t is well defined and adapted to the filtration $\mathcal{F}_t$. We are now ready to formulate our main result.

Theorem 1. *The process and $\{Y_t, \mathcal{F}_t\}$ is a martingale with initial value, $Y_0 = X_0 + f'(Z' - I)e_{s_0}$ and its increments are given by $\Delta Y_t' = e_{s_t}'Zf - e_{s_{t-1}}'PZf$. Moreover, we have*

$$X_t = Y_t + \mu_t + U_t, \tag{1}$$

where $U_t' = e_{s_t}'(I - Z)f$ is a bounded, stationary, and ergodic process with mean zero.

All terms of the expression are given in closed-form, analogous to the Granger representation theorem by Hansen (2005).

It can be shown that ΔY_t is a Markov process with S^{k+1} possible states values. Analogous to P and f, let Q and g denote the transition matrix for ΔY_t and its matrix of state values, respectively. The martingale property dictates that $Qg = 0 \in \mathbb{R}^{S^{k+1}\times d}$. Note that ΔY_t is typically conditionally heterogeneous, as Q is not a matrix of rank one, which would be the structure corresponding to the case where ΔY_t is independent and identically distributed.

The autocovariance structure of the terms in the martingale decomposition is stated next.

Theorem 2. *We have* $\mathrm{var}(\Delta Y_t) = f'Z'(\Lambda_\pi - P'\Lambda_\pi P)Zf$ *where* $\Lambda_\pi = \mathrm{diag}(\pi_1, \ldots, \pi_{S^k})$ *and*

$$\begin{aligned}\mathrm{cov}(U_t, U_{t+j}) &= f'(I - Z)'\Lambda_\pi P^{|j|}(I - Z)f\\ &= f'Z'P'\Lambda_\pi P(P^{|j|} - \Pi)Zf,\end{aligned}$$

[2] The matrix, $I - P + \Pi$, is invertible since the largest eigenvalue of $P - \Pi$ is less than one under Assumption 1.

and the cross correlations are

$$\mathrm{cov}(\Delta Y_t, U_{t+j}) = f'Z'(-\Lambda_\pi + P'\Lambda_\pi P)P^{j+1}Zf, \quad \text{for } j \geq 0,$$

and $\mathrm{cov}(\Delta Y_t, U_{t+j}) = 0$ *for* $j < 0$.

The Theorem shows that the stationary component, U_t, is autocorrelated and, in general, correlated with current and past (but not future) increments, ΔY_t, of the martingale. In the context of financial high-frequency data, where U_t is labelled market microstructure noise, these features are referred to as serially dependent and endogenous noise, that are common empirical characteristics of high-frequency data, see Hansen and Lunde (2006). Let λ_2 denote the second-largest eigenvalue in absolute value of P. Since, $\|P^j - \Pi\| = O(|\lambda_2|^j)$ and $|\lambda_2| < 1$ under Assumption 1, it follows that the autocovariances of U_t decay to zero at an exponential rate.

A corollary to Theorem 2 is that the following.

Corollary 1. *The variance of the observed increments,* $\mathrm{var}(\Delta X_t) = f'(\Lambda_\pi - \pi\pi')f$ *equals*

$$\begin{aligned}&\mathrm{var}(\Delta Y_t) + 2\mathrm{var}(U_t) - \mathrm{cov}(U_{t-1}, U_t) - \mathrm{cov}(U_t, U_{t-1})\\&\quad + \mathrm{cov}(\Delta Y_t, U_t) + \mathrm{cov}(U_t, \Delta Y_t)\\&= f'Z'(I-P)'\Lambda_\pi(I-P)Zf.\end{aligned}$$

3. Concluding remarks and extensions

The martingale decomposition of X_t has several applications, as is the case for the Beveridge–Nelson decomposition for ARIMA processes. In the context of macro time series Y_t and U_t might be labelled the (stochastic) trend and cycle, respectively. In the context of financial high frequency prices, Y_t and U_t could be labelled the efficient prices and market microstructure noise, respectively. In that context, both Y_t and U_t are of separate interest. Moreover, extracting the martingale component, Y_t, offers a motivation for the Markov chain-based estimator of the quadratic variation as in Hansen and Horel (2009). Their estimator is deduced from the long-run variance of X_t, that facilitates a central limit theory and readily available standard errors.

To conclude, we will discuss extensions of the martingale decomposition to accommodate the cases with an infinite number of states (countable), jumps, and inhomogeneous processes.

Suppose that the number of state values for ΔX_t is countable infinite. Then the number of Markov states for $\Delta\mathcal{X}_t$ is countable infinite, and the Markov process can be characterized by $P_{r,s},\ r,s = 1,2,\ldots$. The concept of ergodicity is well defined, and entails a unique stationary distribution, π, that satisfies $\pi_s = \sum_{r=1}^{\infty} P_{r,s}\pi_r$. With $[P^2]_{r,s} = \sum_{j=1}^{\infty} P_{r,j}P_{j,s}$ and higher moments defined similarly, we can define

$$Z_{r,s} = I_{r,s} + \lim_{h\to\infty}\sum_{j=1}^{h}([P^j]_{r,s} - \pi_s),$$

that are well defined provided that the Markov chain is ergodic. It can now be verified that the expressions in Theorems 1 and 2 continue to be applicable to this case.

In financial time series the increments, ΔX_t, are often concentrated about zero, with occasional large changes that are labelled as jumps, see e.g. Huang and Tauchen (2005) and Li (2013). Because jumps are prevalent in high-frequency financial data, the modelling of these data often entails a jump component. One can adapt the martingale decomposition (1) to include a jump component, J_t. This requires a procedure for classifying large increments as jumps and one can then proceed by removing these jumps, e.g. using methods similar to those proposed in Mancini (2009) or Andersen et al. (2012), and then model the remaining returns by the Markov chain methods, to arrive at

$$X_t = Y_t + J_t + \mu_t + U_t,$$

where $J_t = J_{t-1} + \Delta X_t\delta_j$, $\mu_t = \mu_{t-1} + \mu(1-\delta_t)$, $U_t' = (1-\delta_t)e_{s_t}'(I-Z)f$, with δ_t being the indicator for the jumps.

The case with an inhomogeneous Markov chain is theoretically straightforward provided that the transition matrix, $P_{r,s}(t) = \Pr(\Delta\mathcal{X}_t = \mathbf{x}_s|\Delta\mathcal{X}_{t-1} = \mathbf{x}_r)$, satisfies the ergodicity conditions for all t. From the time-varying transition matrix, $P(t)$, one can deduce the increments ΔY_t and $\Delta\mu_t$, as well as U_t, that all depend on $P(t)$. A decomposition arises by piecing the terms together, i.e. $Y_t = Y_0 + \sum_{j=1}^{t}\Delta Y_t$, and again Y_t can be verified to be a martingale, and similarly for other terms. A challenge to implementing this in practice will be to estimate $P(t)$ with a suitable degree of accuracy. This may be achieved by assuming that P is locally homogeneous (piecewise constant), or by imposing a parsimonious structure for the dynamics of $P(t)$, similar to that in the models by Hausman et al. (1992) and Russell and Engle (2005), that can induce an inhomogeneous Markov chain for high-frequency returns.

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Appendix. Proofs

Lemma A.1. *Suppose that Assumption 1 holds.*

(i) $(P-\Pi)^j = P^j - \Pi$,
(ii) $\lim_{h\to\infty}\sum_{j=1}^{h}(P-\Pi)^j = Z - I$, *where* $Z = (I - P + \Pi)^{-1}$,
(iii) $Z\iota = \iota$, $\pi'Z = \pi'$, *and* $PZ = ZP = Z - I + \Pi$,
(iv) $Z - I = (P-\Pi)Z$.

Parts of Lemma A.1 are well known, for instance parts (i) and (ii) are in Brémaud (1999, Chapter 6). For the sake of completeness, we include the (short) proofs of all four parts of the lemma.

Proof. We prove (i) by induction. The identity is obvious for $j = 1$. Now suppose that the identity holds for j. Then

$$\begin{aligned}(P-\Pi)^{j+1} &= (P-\Pi)(P^j - \Pi)\\&= P^{j+1} - \Pi P^j + \Pi^2 - P\Pi = P^{j+1} - \Pi,\end{aligned}$$

where the last identity follows from $\Pi P^j = \Pi^2 = P\Pi = \Pi$.
(ii) Since the chain is ergodic we have $\|P - \Pi\| < 1$, so that P^h converges to Π with $\left\|P^h - \Pi\right\| = O(|\lambda_2|^h)$, where λ_2 is the second largest eigenvalue of P. It follows that $\sum_{j=1}^{\infty}(P^j - \Pi) = \sum_{j=1}^{\infty}(P - \Pi)^j$ is absolutely convergent with $\sum_{j=1}^{\infty}(P-\Pi)^j = \sum_{j=0}^{\infty}(P-\Pi)^j - I = (I - (P-\Pi))^{-1} - I = Z - I$.
(iii) $P^j\iota = \iota$ and $\pi'P^j = \pi'$ for any $j \in \mathbb{N}$; and $\Pi\iota = \iota$ and $\pi'\Pi = \pi'$, so that have $(P^j - \Pi)\iota = \pi'(P^j - \Pi) = 0$. The first two results follow from $Z = I + \sum_{j=1}^{\infty}(P^j - \Pi)$. Next, $PZ = ZP = P + \sum_{j=1}^{\infty}(P^{j+1} - \Pi)$ and

$$\begin{aligned}P + \sum_{j=1}^{\infty}(P^{j+1} - \Pi) &= P + \sum_{j=0}^{\infty}(P^{j+1} - \Pi) - P + \Pi\\&= \sum_{j=1}^{\infty}(P^j - \Pi) + \Pi = Z - I + \Pi.\end{aligned}$$

Finally, the last result follows from $(Z - I) = (I - Z^{-1})Z = (I - I + P - \Pi)Z = (P - \Pi)Z$. □

Proof of Theorem 1. We have $\mathrm{E}(\Delta X'_{t+h}|\Delta\mathcal{X}_t = \mathbf{x}_{s_t}) = e'_{s_t}P^h f$. So with $\Delta\mathcal{X}_t = \mathbf{x}_{s_t}$ we have

$$\begin{aligned}\mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t) &= X_t - \mu_t + \sum_{j=1}^{h}\mathrm{E}(\Delta X_{t+j} - \mu|\mathcal{F}_t)\\ &= X_t - \mu_t + f'\sum_{j=1}^{h}(P^j - \Pi)'e_{s_t},\end{aligned}$$

where the last term is such that $e'_s\sum_{j=1}^{h}(P^j - \Pi)f \to e'_s(Z - I)f$ as $h \to \infty$ by Lemma A.1(ii). Hence,

$$Y_t = \lim_{h\to\infty}\mathrm{E}(X_{t+h} - \mu_{t+h}|\mathcal{F}_t) = X_t - \mu_t + f'(Z - I)'e_{s_t},$$

so that $Y_0 = X_0 + f'(Z - I)'e_{s_0}$ and the increments are given by

$$\begin{aligned}\Delta Y'_t &= \Delta X'_t - \mu' + e'_{s_t}(Z - I)f - e'_{s_{t-1}}(Z - I)f\\ &= e'_{s_t}Zf - e'_{s_{t-1}}(Z + \Pi - I)f = e'_{s_t}Zf - e'_{s_{t-1}}PZf,\end{aligned}$$

where we used Lemma A.1(iii).
This establishes the decomposition, $X_t = Y_t + \mu_t + U_t$, where $U'_t = e'_{s_t}(I - Z)f$. Since U_t is a simple function of $\Delta\mathcal{X}_t$ it follows that U_t is a stationary, ergodic, and bounded process. $\mathrm{E}(U_t) = 0$ follows from $\mathrm{E}(U'_t) = \sum\pi_s e'_s(I - Z)f = (\pi' - \pi'Z)f = 0$, where we used Lemma A.1(iii).

Moreover, $\{Y_t, \mathcal{F}_t\}$ is a martingale, because $Y_t \in \mathcal{F}_t$ and

$$\begin{aligned}\mathrm{E}(e'_sZf - e'_rPZf|\Delta\mathcal{X}_{t-1} = \mathbf{x}_r) &= \sum_s P_{r,s}e'_sZf - e'_rPZf\\ &= e'_rPZf - e'_rPZf = 0,\end{aligned}$$

for any $r = 1, \ldots, S^k$, where r and s are short for s_{t-1} and s_t, respectively (defined by $\Delta\mathcal{X}_{t-1} = \mathbf{x}_r$ and $\Delta\mathcal{X}_t = \mathbf{x}_s$). □

In the proof of Theorem 2 we use the following identities

$$\begin{aligned}&\sum_{r,s}\pi_rP_{r,s}e_re'_r = \sum_{r,s}\pi_rP_{r,s}e_se'_s = \Lambda_\pi, \quad \text{and}\\ &\sum_{r,s}\pi_r[P^j]_{r,s}e_re'_s = \Lambda_\pi P^j,\end{aligned} \tag{A.1}$$

that are easily verified.

Proof of Theorem 2. For the variance of the martingale increments we have

$$\begin{aligned}&\mathrm{E}(\Delta Y_t\Delta Y'_t) = \mathrm{E}[(f'Z'e_{s_t} - f'Z'P'e_{s_{t-1}})(e'_{s_t}Zf - e'_{s_{t-1}}PZf)],\\ &= \sum_{r,s}\pi_rP_{r,s}f'Z'(e_s - P'e_r)(e'_s - e'_rP)Zf\\ &= \sum_{r,s}\pi_rP_{r,s}f'Z'(e_se'_s - e_se'_rP - P'e_re'_s + P'e_re'_rP)Zf\\ &= f'Z'(\Lambda_\pi - P'\Lambda_\pi P - P'\Lambda_\pi P + P'\Lambda_\pi P)Zf\\ &= f'Z'(\Lambda_\pi - P'\Lambda_\pi P)Zf,\end{aligned}$$

where we used (A.1) in the second last equality.
Concerning the stationary component of the decomposition we have for $j \geq 0$ that

$$\begin{aligned}\mathrm{E}(U_tU'_{t+j}) &= \mathrm{E}[f'(I - Z)'e_{s_t}e'_{s_{t+j}}(I - Z)f]\\ &= \sum_{r,s}\pi_r[P^j]_{r,s}f'(I - Z)'e_re'_s(I - Z)f\\ &= f'(I - Z)'\Lambda_\pi P^j(I - Z)f\\ &= f'Z'(\Pi - P)'\Lambda_\pi P^j(\Pi - P)Zf\\ &= f'Z'P'\Lambda_\pi P(P^j - \Pi)Zf,\end{aligned}$$

where we used Lemma A.1(iv) in the second last equality.

Finally, for the cross covariance we first note that,

$$\begin{aligned}&\sum_{r,s,v}\pi_rP_{r,s}[P^j]_{s,v}e_se'_v = \sum_{s,v}\pi_s[P^j]_{s,v}e_se'_v = \Lambda_\pi P^j,\\ &\sum_{r,s,v}\pi_rP_{r,s}[P^j]_{s,v}e_re'_v = \sum_{r,s,v}\pi_r[P^{j+1}]_{r,v}e_re'_v = \Lambda_\pi P^{j+1},\end{aligned}$$

where the first identities in the two equations follow by $\sum_r\pi_rP_{r,s} = \pi_s$ and $\sum_s P_{r,s}[P^j]_{s,v} = [P^{j+1}]_{r,v}$, respectively, and the last equalities both follow from the last variant of (A.1). So for $j \geq 0$ we have

$$\begin{aligned}\mathrm{E}(\Delta Y_tU'_{t+j}) &= \mathrm{E}[(e'_{s_t}Zf - e'_{s_{t-1}}PZf)'e'_{s_{t+j}}(I - Z)f]\\ &= \sum_{r,s,v}\pi_rP_{r,s}[P^j]_{s,v}f'Z'(e_s - P'e_r)e'_v(\Pi - P)Zf\\ &= f'Z'[\Lambda_\pi P^j(\Pi - P) - P'\Lambda_\pi P^{j+1}(\Pi - P)]Zf\\ &= f'Z'[\pi\pi' - \Lambda_\pi P^{j+1} - \pi\pi' + P'\Lambda_\pi P^{j+2}]Zf\\ &= f'Z'(-\Lambda_\pi + P'\Lambda_\pi P)P^{j+1}Zf.\end{aligned}$$

That $\mathrm{E}(\Delta Y_tU'_{t+j}) = 0$ for $j < 0$ can be verified similarly. However, this is not required because the zero covariances are a simple consequence of martingale property of Y_t that was established in the proof of Theorem 1. □

Proof of Corollary 1. By substituting the expressions from Theorem 2 and using $\mathrm{cov}(\Delta Y_t, U_{t-1}) = 0$, one finds that the expression in Corollary 1 equals $f'Z'AZf$, where

$$\begin{aligned}A &= (\Lambda_\pi - P'\Lambda_\pi P) + 2P'\Lambda_\pi P(I - \Pi) - P'\Lambda_\pi P(P - \Pi)\\ &\quad - (P - \Pi)'P'\Lambda_\pi P + (-\Lambda_\pi + P'\Lambda_\pi P)P\\ &\quad + P'(-\Lambda_\pi + P'\Lambda_\pi P)\\ &= \Lambda_\pi + P'\Lambda_\pi P - 2\pi\pi' + \pi\pi' + \pi\pi' - \Lambda_\pi P - P'\Lambda_\pi\\ &= (I - P)'\Lambda_\pi(I - P),\end{aligned}$$

which proves the equality in the corollary. That $f'Z'AZf = f'(\Lambda_\pi - \pi\pi')f$ follows from

$$\begin{aligned}(I - P)'\Lambda_\pi(I - P) &= (I - P + \Pi)'\Lambda_\pi(I - P + \Pi) - \pi\pi'\\ &= (I - P + \Pi)'(\Lambda_\pi - \pi\pi')(I - P + \Pi),\end{aligned}$$

which equals $(Z^{-1})'(\Lambda_\pi - \pi\pi')Z^{-1}$. This completes the proof. □

References

Adda, J., Cooper, R., 2000. Balladurette and juppette: A discrete analysis of scrapping subsidies. J. Polit. Econ. 108, 778–806.
Andersen, T., Dobrev, D., Schaumburg, E., 2012. Jump-robust volatility estimation using nearest neighbor truncation. J. Econometrics 169, 75–93.
Beveridge, S., Nelson, C.R., 1981. A new approach to decompositions of time series into permanent and transitory components with particular attentions to measurement of the 'business cycle'. J. Monetary Econ. 7, 151–174.
Brémaud, P., 1999. Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues. Springer.
Fuller, W.A., 1976. Introduction to Statistical Time Series. Wiley, New York.
Gordin, M.I., 1969. The central limit theorem for stationary processes. Sov. Math. Dokl. 10, 1174–1176.
Hamilton, J.D., 1994. Time Series Analysis. Princeton University Press, Princeton, NJ.
Hansen, P.R., 2005. Granger's representation theorem: A closed-form expression for $I(1)$ processes. Econom. J. 8, 23–38.
Hansen, L.P., 2012. Dynamic valuation decomposition within stochastic economies. Econometrica 80, 911–967.
Hansen, P.R., Horel, G., 2009. Quadratic variation by Markov chains, Working Paper.
Hansen, P.R., Lunde, A., 2006. Realized variance and market microstructure noise. J. Bus. Econom. Statist. 24, 127–218. The 2005 Invited Address with Comments and Rejoinder.
Harvey, A.C., Koopman, S.J., 2002. Signal extraction and the formulation of unobserved components models. Econom. J. 3, 84–107.
Hasbrouck, J., 1995. One security, many markets: determining the contributions to price discovery. J. Finance 50, 1175–1198.
Hausman, J.A., Lo, A.W., MacKinlay, A.C., 1992. An ordered probit analysis of transaction stock prices. J. Financ. Econ. 31, 319–379.
Huang, X., Tauchen, G., 2005. The relative contribution of jumps to total price variation. J. Financ. Econ. 3, 456–499.

Johansen, S., 1991. Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. Econometrica 59, 1551–1580.
Large, J., 2011. Estimating quadratic variation when quoted prices change by a constant increment. J. Econometrics 160, 2–11.
Li, J., 2013. Robust estimation and inference for jumps in noisy high frequency data: a local-to-continuity theory for the pre-averaging method. Econometrica 81, 1673–1693.
Low, C.N., Anderson, H.M., 2008. Economic applications: The Beveridge–Nelson decomposition. In: Hyndman, R.J., Koehler, A.B., Ord, J.K., Snyder, R.D. (Eds.), Forecasting with Exponential Smoothing. Springer, pp. 325–337 (Chapter 20).
Mancini, C., 2009. Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. Scand. J. Statist. 36 (2), 270–296.
Phillips, P.C.B., Solo, V., 1992. Asymptotics for linear processes. Ann. Statist. 20, 971–1001.
Russell, J.R., Engle, R.F., 2005. A discrete-state continuous-time model of financial transactions prices and times: The autoregressive conditional multinomial-autoregressive conditional duration model. J. Bus. Econom. Statist. 23, 166–180.
Tauchen, G., 1986. Finite state Markov-chain approximations to univariate and vector autoregressions. Econom. Lett. 20, 177–181.