Exponential GARCH Modeling With Realized Measures of Volatility

Peter Reinhard Hansen
European University Institute, 50014 San Domenico di Fiesole FI, Italy, University of North Carolina, Chapel Hill, NC, USA, and CREATES, Aarhus University, 8000 Aarhus C, Denmark (*peter.hansen@eui.eu*)

Zhuo Huang
National School of Development, China Center for Economic Research, Peking University, Beijing, China (*zhuohuang@nsd.pku.edu.cn*)

We introduce the realized exponential GARCH model that can use multiple realized volatility measures for the modeling of a return series. The model specifies the dynamic properties of both returns and realized measures, and is characterized by a flexible modeling of the dependence between returns and volatility. We apply the model to 27 stocks and an exchange traded fund that tracks the S&P 500 index and find specifications with multiple realized measures that dominate those that rely on a single realized measure. The empirical analysis suggests some convenient simplifications and highlights the advantages of the new specification.

KEY WORDS: EGARCH; High-frequency data; Leverage effect; Realized variance.

1. INTRODUCTION

The realized GARCH framework by Hansen, Huang, and Shek (2012) provides a structure for the joint modeling of returns $\{r_t\}$ and realized measures of volatility $\{x_t\}$. In this article, we introduce a new variant within this framework, called the realized exponential generalized auto regressive conditional (GARCH) model. Key features of this model include: (i) The ability to incorporate multiple realized measures of volatility, such as the realized variance and the daily range; (ii) A flexible modeling of the dependence between returns and volatility, which is known to be empirically important—a result that is confirmed in our empirical analysis of 27 liquid stocks and the exchange traded index fund, SPY. The present article also contributed to the literature with a number of empirical results. We undertake an extensive empirical analysis that motivates several refinements and simplifications of the model. We compare a range of realized measures and show that the log-likelihood for daily returns can be improved by the use of multiple realized measures. This is true in-sample and out-of-sample. The empirical analysis also highlights the advantages of the new specification, and we provide theoretical insight about the underlying reasons for this.

GARCH models are time-series models that specify the conditional distribution of the next period's observation—typically a return on some financial asset. The key variable is the conditional variance that is defined by past variables. Conventional GARCH models, such as the auto regressive conditional heteroscedasticity (ARCH) by Engle (1982) and GARCH by Bollerslev (1986), rely exclusively on daily returns (typically squared returns) for the modeling of volatility. A shortcoming of conventional GARCH models is the fact that returns are rather weak signals about the level of volatility. This makes GARCH models poorly suited for situations where volatility "jumps" to a new level over a short period of time. In such a situation, a GARCH model will be slow at "catching up," so that it takes several periods for the conditional variance (implied by the GARCH model) to reach the new level, see Andersen et al. (2003) for discussion on this. Incorporating realized measures into GARCH models can greatly alleviate this problem.

A wide range of realized measures of volatility has been proposed in the literature since Andersen and Bollerslev (1998) showed that such measures can be very useful for the evaluation of volatility models. Realized measures of volatility, such as the popular realized variance, are computed from high-frequency data, see Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2002). The realized variance is sensitive to market microstructure noise, which has motivated the development of robust realized measures, such as the two-scale and multi-scale estimator by Zhang, Mykland, and Aït-Sahalia (2005) and Zhang (2006), respectively, the realized kernel by Barndorff-Nielsen et al. (2008), the realized range by Christensen and Podolskij (2007), see also Andersen, Dobrev, and Schaumburg (2008), Hansen and Horel (2009), and references therein. Because realized measures are far more informative about the current level of volatility than the squared return, it can be very useful to include such in the modeling of volatility. The economic and statistical gains from incorporating realized measures in volatility models are typically found to be large, see, for example, Christoffersen et al. (2014) and Dobrev and Szerszen (2010). Following the early work by Andersen and Bollerslev (1998) that had documented the value of using realized measures in the evaluation of volatility models, Engle (2002) explored the idea of including the realized variance as a predetermined variable in the GARCH equation, and found it to be highly significant and greatly enhancing the empirical fit, see also Forsberg and Bollerslev (2002). The first *complete* model (complete in

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the sense of specifying the dynamic properties of all observed time-series) was introduced by Engle and Gallo (2006), who referred to the model as a multiplicative error model (MEM). The MEM framework operates with multiple latent volatility processes—one for returns and one for each of the realized measures. This structure is also the basis for the model proposed by Shephard and Sheppard (2010), who referred to their model as the HEAVY model. See also Visser (2011) and Chen, Ghysels, and Wang (2011).

The realized GARCH framework takes a different approach. Instead of introducing additional latent variables to the model, the realized GARCH framework is based on measurement equations that tie the realized measure to the latent conditional variance. This facilitates an explicit modeling of leverage effect, and circumvents the need for additional latent volatility processes. The idea of using a measurement equation to tie the realized measure to the latent volatility goes back to Takahashi, Omori, and Watanabe (2009), who used it in the context of stochastic volatility models. Additional MEM specifications have been explored and developed in Cipollini, Engle, and Gallo (2009) and Brownless and Gallo (2010).

To illustrate the structure of a realized EGARCH model and how it compares with conventional models, we give a brief preview of our empirical result. Below we have estimated the GARCH(1,1) by Bollerslev (1986), the EGARCH(1,1) by Nelson (1991), a slightly modified EGARCH, and a realized EGARCH model with daily returns, $\{r_t\}$, on the S&P500 index over the sample period spanning January 1, 2002 to December 31, 2013. The full details will be presented in Section 4. These four models have the same *return equation*, $r_t = \mu + \sqrt{h_t} z_t$ with $z_t \sim \mathrm{iid}N(0,1)$, but their specifications for dynamic properties of the conditional variance, $h_t = \mathrm{var}(r_t|\mathcal{F}_{t-1})$, differ in important ways. For the GARCH(1,1), the EGARCH(1,1), and a variation of the latter, we estimated their GARCH equations to

$$h_{t+1} = +\underset{(0.003)}{0.016} + \underset{(0.010)}{0.987} h_t + \underset{(0.009)}{0.085}\left(r_t^2 - h_t\right),$$

$$\log h_{t+1} = +\underset{(0.002)}{0.014} + \underset{(0.003)}{0.980} \log h_t - \underset{(0.012)}{0.136} z_t + \underset{(0.013)}{0.110}\left(|z_t| - \sqrt{\frac{2}{\pi}}\right),$$

$$\log h_{t+1} = -\underset{(0.001)}{0.002} + \underset{(0.003)}{0.978} \log h_t - \underset{(0.012)}{0.132} z_t + \underset{(0.005)}{0.036}\left(z_t^2 - 1\right),$$

respectively, where the numbers in brackets are standard error. In comparison, a realized EGARCH model that uses two realized measures leads to the following estimated GARCH equation:

$$\log h_{t+1} = -\underset{(0.004)}{0.005} + \underset{(0.003)}{0.967} \log h_t - \underset{(0.007)}{0.147} z_t + \underset{(0.003)}{0.0259}\left(z_t^2 - 1\right) + \underset{(0.019)}{0.278} u_{\mathrm{RK},t} + \underset{(0.010)}{0.070} u_{\mathrm{DR},t}.$$

The additional variables on the right-hand side, $u_{\mathrm{RK},t}$ and $u_{\mathrm{DR},t}$, are given from the two (estimated) measurement equations:

$$\log x_{\mathrm{RK},t} = -\underset{(0.026)}{0.525} + \log h_t - \underset{(0.007)}{0.137} z_t + \underset{(0.004)}{0.028}\left(z_t^2 - 1\right) + u_{\mathrm{RK},t},$$

$$\log x_{\mathrm{DR},t} = +\underset{(0.024)}{0.437} + \log h_t - \underset{(0.014)}{0.075} z_t + \underset{(0.009)}{0.190}\left(z_t^2 - 1\right) + u_{\mathrm{DR},t},$$

which relate the realized measures, $x_{\mathrm{RK},t}$ and $x_{\mathrm{DR},t}$, to the conditional variance. The realized measures used here, $x_{\mathrm{RK},t}$ and $x_{\mathrm{DR},t}$, are the realized kernel for day t and the daily (squared) range, respectively. In the likelihood analysis, the bivariate vector, $(u_{\mathrm{RK}}, u_{\mathrm{DR}})'$, is specified to be iid and Gaussian distributed with mean zero and a variance-covariance matrix, which is estimated to be

$$\hat{\Sigma} = \begin{pmatrix} 0.130 & 0.139 \\ 0.139 & 0.418 \end{pmatrix}.$$

The realized measures contribute to modeling the volatility dynamics through the coefficients for u_{RK} and u_{DR} in the GARCH equations. Both coefficients are significant but the realized kernel is evidently more important for the volatility dynamics. The structure of the estimated covariance matrix, $\hat{\Sigma}$, shows (not surprisingly) that the realized kernel is a far more accurate measurement of the conditional variance than is the daily range. The fact that the covariance is close to the variance of $u_{\mathrm{RK},t}$ suggest that $u_{\mathrm{DR},t} \simeq u_{\mathrm{RK},t} + \epsilon_t$, where ϵ_t is uncorrelated with $u_{\mathrm{RK},t}$.

The real benefits of including realized measures in the GARCH modeling are revealed by comparing the value of the log-likelihood function for daily returns. Note that the likelihood for returns only constitutes one component of the likelihood that is being maximized by the realized EGARCH model. The full likelihood for the realized EGARCH model also includes the likelihood related to the realized measures.

In Table 1, we present the value of the log-likelihood functions for four specifications. The first column presents the value of the log-likelihood for returns over the full sample period. The following two pairs of columns present in-sample and out-of-sample log-likelihoods using two different sample splits. The out-of-sample log-likelihood is simply the log-likelihood function computed with the out-of-sample data, but using the parameter estimates from the corresponding in-sample period. The realized EGARCH model strongly dominates the conventional GARCH models that rely exclusively on daily returns. Even though the realized EGARCH model is maximizing a likelihood for $(r_t, x_{\mathrm{RK},t}, x_{\mathrm{DR},t})'$, it does produce a much higher value of the partial log-likelihood than the conventional GARCH models. Out-of-sample we observe an even larger difference between the log-likelihoods, as the realized EGARCH is 40–60 log-likelihood points better than all of the conventional GARCH specifications. This demonstrates the benefits of incorporating realized measures in models for return volatility.

This article is organized as follows. We introduce the realized exponential GARCH model in Section 2, and quasi-maximum likelihood estimation (QMLE) and inference are discussed in Section 3. Our empirical results are mainly presented in Section 4, and Section 5 compares the proposed specification to that in Hansen, Huang, and Shek (2012) in terms of both theoretical and empirical properties. We make some concluding remarks in Section 6. Proofs are given in Appendix A, Appendix B has some supporting empirical results, and Appendix C presents results for a generalized model where the innovations in the measurement equation, u_t, are modeled with ARCH and GARCH features.

Table 1. Log-likelihoods (for returns) for each of the models are reported for different sample periods, including two out-of-sample periods. The full sample results in the first column are based on the period January 1, 2002, to December 31, 2013. The two subsequent pairs of columns are in-sample and out-of-sample results based on a sample split $\frac{1}{2}$ and $\frac{2}{3}$ into the sample. The EGARCH is the original specification proposed in Nelson (1991), the quadratic variant of the EGARCH performs similarly, but has the same leverage function as the realized EGARCH model

	Full-sample	In-sample	Out-of-sample	In-sample	Out-of-sample
	2002–2013	2002–2007	2008–2013	2002–2009	2010–2013
GARCH	−4255.02	−1919.23	−2344.21	−2900.98	−1356.73
EGARCH	−4183.69	−1888.65	−2312.13	−2859.58	−1329.02
EGARCH (quadratic)	−4188.24	−1892.20	−2311.86	−2863.38	−1330.92
Realized EGARCH	−4120.71	−1878.96	−2251.14	−2833.63	−1290.63

2. REALIZED EGARCH MODEL

In this section, we introduce the realized exponential GARCH model (in short, realized EGARCH). We use this terminology because the model shares some features with the EGARCH model by Nelson (1991). The realized EGARCH model is well suited for the case with multiple realized measures of volatility, in which case we let x_t denote a vector of realized measures, $x_t = (x_{1,t}, \ldots, x_{K,t})'$. Moreover, the model permits a more flexible modeling of the joint dependence of returns and volatility. The latter is shown to be very useful in our empirical analysis. The vector of realized measures, x_t, may include the realized variance, bipower variation, daily range, squared return, and robust measures such as the realized kernel.

Let $\{\mathcal{F}_t\}$ be a filtration so that (r_t, x_t) is adapted to $\mathcal{F}_t$, and define the conditional mean, $\mu_t = \mathrm{E}(r_t|\mathcal{F}_{t-1})$, and conditional variance, $h_t = \mathrm{var}(r_t|\mathcal{F}_{t-1})$. The realized EGARCH model with K realized measures is given by the following equations:

$$\begin{aligned} r_t &= \mu_t + \sqrt{h_t} z_t, \\ \log h_t &= \omega + \beta \log h_{t-1} + \tau(z_{t-1}) + \gamma' u_{t-1}, \\ \log x_{k,t} &= \xi_k + \varphi_k \log h_t + \delta_{(k)}(z_t) + u_{k,t}, \qquad k = 1, \ldots, K. \end{aligned}$$

We refer to these as the *return equation*, the *GARCH equation*, and the *measurement equation(s)*, respectively. We discuss these three equations in greater details below. In our quasi-likelihood analysis, we adopt a Gaussian specification, $z_t \sim N(0, 1)$ and $u_t \sim N(0, \Sigma)$, where z_t and $u_t = (u_{1,t}, \ldots, u_{K,t})'$ are mutually and serially independent. The leverage functions, $\tau(z)$ and $\delta_{(k)}(z)$, $k = 1, \ldots, K$, play an important role to make the independence between z_t and u_t realistic in practice. In the empirical analysis, we adopt quadratic form for the leverage functions,

$$\tau(z) = \tau_1 z + \tau_2(z^2 - 1)$$
$$\text{and } \delta_{(k)}(z) = \delta_{k,1} z + \delta_{k,2}(z^2 - 1), \quad k = 1, \ldots, K.$$

The leverage functions facilitate a modeling of the dependence between return shocks and volatility shocks, which is empirically important, and the volatility shock, $v_t = \mathrm{E}(\log h_{t+1}|\mathcal{F}_t) - \mathrm{E}(\log h_{t+1}|\mathcal{F}_{t-1})$, is given by $v_t = \tau(z_t) + \gamma' u_t$ in this model.

The return equation is standard in GARCH models. The conditional mean, μ_t, may be modeled with a GARCH-in-mean specification or simply a constant. In fact, imposing the constraint $\mu_t = 0$ can result in better out-of-sample fit relative to a model based on an unrestricted μ. That is indeed what we find in our empirical analysis.

The GARCH equation plays a central role in models of the conditional variance, and a key feature of the realized EGARCH model is the presence of a leverage function, $\tau(z_{t-1})$, in the GARCH equation. The realized GARCH model in Hansen, Huang, and Shek (2012) only includes a leverage function in the measurement equation. The EGARCH model is often based on $\tau(z) = az + b|z|$. We prefer a polynomial specification for τ for empirical reasons and because the likelihood analysis is simplified by the fact that $\tau(z)$ is differentiable at zero. Another key feature of the realized EGARCH model is the last term in the GARCH equation. This term, $\gamma' u_{t-1}$, is the main channel by which the realized measures drive expectations of future volatility up or down. The fact that u_t is K-dimensional enables us to use multiple realized measures of volatility. It is worth noting that this GARCH equation specifies an AR(1) model for the conditional variance, with innovations given by $\tau(z_{t-1}) + \gamma' u_{t-1}$. Hence, the parameter β summarizes the persistence of volatility, whereas γ represents how informative the realized measures are about future volatility.

The measurement equation defines the link between the (ex-post) realized measures of volatility and the (ex-ante) conditional variance. An ex-post measure of volatility will differ from the conditional variance for a number of reasons. One source for this discrepancy is because realized measures are not perfect measures of volatility. Empirical measures entail sampling error and even the most accurate realized measures are known to have nonnegligible sampling error in practice. Another source for the discrepancy is due to difference between ex-post volatility and ex-ante volatility, which we can label the volatility shock.

The three equations fully characterize the dynamic properties of returns and realized measures of volatility. So that the model is complete in the sense that it fully specifies the dynamic properties of both returns and the realized measures.

3. ESTIMATION AND INFERENCE

In this section, we discuss estimation and inference within the quasi-maximum likelihood framework. The analysis largely follows that in Hansen, Huang, and Shek (2012), but the fact that the present framework allows for multiple realized measures and the introduction of a leverage function in the GARCH equation requires some modifications and extensions of the analysis in Hansen, Huang, and Shek (2012).

We adopt a Gaussian specification by assuming $z_t \sim \mathrm{iid}N(0, 1)$ and $u_t \sim \mathrm{iid}N(0, \Sigma)$, with z_t and u_t independent. Express the leverage functions as $\tau(z_t) = \tau' a(z_t)$ and $\delta_{(k)}(z_t) =$

$\delta_k' b(z_t)$, $k = 1, \ldots, K$, where $a(z_t)$ and $b(z_t)$ are known functions of z_t. In our empirical analysis, we use $a_t = b_t = (z_t, z_t^2 - 1)'$. The initial value of the conditional variance, h_1, is treated as an unknown parameter. In Section 3.1, we show that the choice for initial value is asymptotically negligible under reasonable assumptions, a result that is analogous to that for conventional GARCH models, see, for example, Francq and Zakoian (2010, chap. 7).

So the parameters in the model are

$$\theta = \left(\tilde{h}_1, \mu, \lambda', \psi_1', \ldots, \psi_K'\right)' \qquad \text{and} \qquad \Sigma,$$

where

$$\lambda = (\omega, \beta, \tau', \gamma')' \quad \text{and} \quad \psi_k = \left(\xi_k, \varphi_k, \delta_k'\right)', \quad k = 1, \ldots, K.$$

To simplify the notation, we write $\tilde{h}_t = \log h_t$ and $\tilde{x}_{k,t} = \log x_{k,t}$, and define

$$g_t = \left(1, \tilde{h}_t, a_t', u_t'\right)', \qquad \text{and} \qquad m_t = \left(1, \tilde{h}_t, b_t'\right)',$$

where $a_t = a(z_t)$ and $b_t = b(z_t)$. This enables us to express the GARCH and measurement equations as

$$\tilde{h}_t = \lambda' g_{t-1} \qquad \text{and} \qquad \tilde{x}_{k,t} = \psi_k' m_t + u_{k,t}, \quad k = 1, \ldots, K,$$

respectively.

The quasi-log-likelihood function is given by

$$\ell(r, x; \theta, \Sigma_u) = -\frac{1}{2} \sum_{t=1}^{n} \left[\log(2\pi) + \tilde{h}_t + z_t^2 + K \log(2\pi) + \log(|\Sigma|) + u_t' \Sigma^{-1} u_t\right],$$

where $z_t = z_t(\theta) = (r_t - \mu)/\sqrt{h_t}$ and $u_{k,t}(\theta) = \tilde{x}_{k,t} - \xi_k - \varphi_k \tilde{h}_t(\theta) - \delta_k' b(z_t(\theta))$. We estimate the model's parameters by maximizing the quasi-log-likelihood function, $\ell(r, x; \theta, \Sigma)$, with respect to θ and Σ. The log-likelihood function has a convenient structure, so that (partial) maximization with respect to Σ, for a given value of θ, has the simple solution:

$$\hat{\Sigma}(\theta) = \frac{1}{n} \sum_{t=1}^{n} u_t(\theta) u_t(\theta)', \tag{1}$$

where we have made explicit that u_t depends on θ but, importantly, does not depend on the covariance matrix Σ. We can therefore simplify the maximization problem to $\arg\max_\theta \ell(r, x; \theta, \hat{\Sigma}(\theta))$, where

$$\ell(r, x; \theta, \hat{\Sigma}(\theta)) \propto -\frac{1}{2} \sum_{t=1}^{n} \left[\log h_t(\theta) + z_t(\theta)^2\right] - \frac{n}{2} \log\det \hat{\Sigma}(\theta),$$

and where we use the fact that $\sum_{t=1}^n u_t(\theta)' \hat{\Sigma}(\theta)^{-1} u_t(\theta) = \mathrm{tr}\{\sum_{t=1}^n \hat{\Sigma}(\theta)^{-1} u_t(\theta) u_t(\theta)'\} = nK$, which does not depend on θ.

To compute robust standard errors, we need to derive the dynamic properties of the score and hessian. A key component in this dynamics is the derivative of $\log h_{t+1}$ with respect to $\log h_t$, which is stated next.

Lemma 1. Let $\varphi = (\varphi_1, \ldots, \varphi_K)'$ and let D be the matrix whose kth row is δ_k', $k = 1, \ldots, K$. Then $\partial \log h_{t+1} / \partial \log h_t = A(z_t)$ and $-2\partial \ell_t / \partial \log h_t = B(z_t, u_t)$, where

$$A(z_t) = (\beta - \gamma' \varphi) + \frac{1}{2}\left(\gamma' D \dot{b}_{z_t} - \tau' \dot{a}_{z_t}\right) z_t,$$

$$B(z_t, u_t) = \left(1 - z_t^2\right) + u_t' \Sigma^{-1} \left(D \dot{b}_{z_t} z_t - 2\varphi\right),$$

with $\dot{a}_{z_t} = \partial a(z_t)/\partial z_t$ and $\dot{b}_{z_t} = \partial b(z_t)/\partial z_t$.

We obtain the following results for $\dot{h}_{\lambda,t} = \frac{\partial \tilde{h}_t}{\partial \lambda}$ and $\dot{h}_{\mu,t} = \frac{\partial \tilde{h}_t}{\partial \mu}$, which we use to simplify our expressions for the score function.

Lemma 2. $\dot{h}_{\lambda,t} = \frac{\partial \tilde{h}_t}{\partial \lambda}$ and $\dot{h}_{\mu,t} = \frac{\partial \tilde{h}_t}{\partial \mu}$ are given from the stochastic recursions:

$$\dot{h}_{\lambda,t+1} = A(z_t) \dot{h}_{\lambda,t} + g_t,$$

$$\dot{h}_{\mu,t+1} = A(z_t) \dot{h}_{\mu,t} + (\gamma' D \dot{b}_{z_t} - \tau' \dot{a}_{z_t}) h_t^{-\frac{1}{2}},$$

for $t \geq 1$ with $\dot{h}_{\lambda,1} = \dot{h}_{\mu,1} = 0$.

Next we turn to the score that defines the first-order conditions for the quasi-maximum likelihood estimators.

Theorem 1. The scores, $\frac{\partial \ell}{\partial \theta} = \sum_{t=1}^n \frac{\partial \ell_t}{\partial \theta}$ with $\theta = (\tilde{h}_1, \mu, \lambda', \psi_1', \ldots, \psi_K')'$ and $\frac{\partial \ell}{\partial \Sigma^{-1}} = \sum_{t=1}^n \frac{\partial \ell_t}{\partial \Sigma^{-1}}$ are given from

$$\frac{\partial \ell_t}{\partial \theta} = -\frac{1}{2} \begin{pmatrix} B(z_t, u_t) \prod_{s=1}^{t-1} A(z_s) \\ B(z_t, u_t) \dot{h}_{\mu,t} + 2[z_t - u_t' \Sigma^{-1} D \dot{b}_{z_t}] h_t^{-\frac{1}{2}} \\ B(z_t, u_t) \dot{h}_{\lambda,t} \\ -2\Sigma^{-1} u_t \otimes m_t \end{pmatrix}$$

and

$$\frac{\partial \ell_t}{\partial \Sigma^{-1}} = \frac{1}{2}(\Sigma - u_t u_t'). \tag{2}$$

From Lemma 1 and Theorem 1 it follows that

Corollary 1. The score function is a martingale difference process, provided that $\mathrm{E}(z_t|\mathcal{F}_{t-1}) = 0$, $\mathrm{E}(z_t^2|\mathcal{F}_{t-1}) = 1$, $\mathrm{E}(u_t|z_t, \mathcal{F}_{t-1}) = 0$, and $\mathrm{E}(u_t u_t'|\mathcal{F}_{t-1}) = \Sigma$.

The first two conditions, $\mathrm{E}(z_t|\mathcal{F}_{t-1}) = 0$ and $\mathrm{E}(z_t^2|\mathcal{F}_{t-1}) = 1$, translate into the conditional mean and variance for r_t being correctly specified. The third condition, $\mathrm{E}(u_t|z_t, \mathcal{F}_{t-1}) = 0$, is essentially a requirement that $\delta_{(k)}$ is sufficiently flexible to capture the conditional mean: $\mathrm{E}(\log x_{k,t} - \xi_k - \varphi_k \log h_t | z_t, \mathcal{F}_{t-1}) = \mathrm{E}(\log x_{k,t} - \xi_k - \varphi_k \log h_t | z_t) = \delta_{(k)}(z_t)$. The last requirement, $\mathrm{E}(u_t u_t'|\mathcal{F}_{t-1}) = \Sigma$, is a homoscedasticity assumption on u_t. This assumption is in line with the limit theory for realized measures, which shows that their standard errors are roughly proportional to their levels. In the heteroscedastic case where $\mathrm{E}(u_t u_t'|\mathcal{F}_{t-1}) = \Sigma$ does not hold, this would only impact the part of the score that relates to Σ.

The first-order conditions for Σ lead to the closed-form expression in (1), where $\hat{u}_{k,t} = u_{k,t}(\hat{\theta}) = \tilde{x}_{k,t} - \hat{\xi}_k - \hat{\varphi}_k \tilde{h}_t(\hat{\theta}) - \hat{\delta}_k' b(z_t(\hat{\theta}))$, $k = 1, \ldots, K$. This expression reduces the complexity of the optimization problem substantially.

The GARCH equation implies that $\log h_t$ has the stationary MA(∞) representation

$$\log h_t = \beta^j \log h_{t-j} + \sum_{i=0}^{j-1} \beta^j [\tau(z_{t-1-i}) + \gamma' u_{t-1-i}],$$

so that h_t has a stationary representation if $|\beta| < 1$. In the likelihood analysis, however, it is the random variable, $A(z_t) = (\beta - \gamma'\varphi) + \frac{1}{2}(\gamma' D\dot{b}_{z_t} - \tau'\dot{a}_{z_t})z_t$, which shows up as the "autoregressive coefficient" in various expressions. This occurs because the derivative is taken while the observables (r_t, x_t) are held constant, and this phenomenon is well known from the EGARCH model.

3.1 Influence of the Initial Value

We note that the effect that h_1 has on the log-likelihood is proportional to $\sum_{t=1}^{n}\prod_{s=1}^{t-1} A(z_s)$, and that $\prod_{s=1}^{t-1} A(z_s)$ vanishes in probability (exponentially fast) provided that

$$\mathrm{E}\log|A(z_t)| < 0, \tag{3}$$

because $|\prod_s A(z_s)| = \exp n\{\frac{1}{n}\sum_s \log|A(z_s)|\}$ and $\frac{1}{n}\sum_s \log|A(z_s)| \overset{\text{a.s.}}{\to} \mathrm{E}\log|A(z_t)|$ by the law of large numbers. By Jensen's inequality, we observe that $\mathrm{E}|A(z_t)| < 1$ is a sufficient condition for (3). In our empirical analysis, this condition is satisfied in all the models we have estimated. For instance, the estimated model for SPY close-to-close returns with the realized kernel, $x_{RK,t}$, has $\mathrm{E}|A(z_t)| = 0.76$, whereas the estimated model using two realized measures, the realized kernel and the daily range, has $\mathrm{E}|A(z_t)| = 0.835$. (Here the expectation is computed using parameter estimates and a Gaussian specification for z_t.)

3.2 Asymptotic Distribution of Estimators

A drawback of conventional GARCH models is that the asymptotic analysis of estimators and their properties are rather challenging. It took more than two decades to establish several elementary results for some of the simplest models, see Bollerslev and Wooldridge (1992), Lee and Hansen (1994), Lumsdaine (1996), Jensen and Rahbek (2004), Straumann and Mikosch (2006), Kristensen and Rahbek (2005, 2009), and references therein. The asymptotic analysis of realized EGARCH model is similarly complicated, so it is beyond the scope of this article to fully establish the asymptotic theory for the estimators. However, based on the theoretical results in this section, including the martingale difference properties stated in Corollary 1, it seems reasonable to conjecture the following about the limit distribution, which we use to compute standard errors in our empirical analysis.

Conjecture 1. The QMLE estimators

$$\sqrt{n}\begin{pmatrix}\hat{\theta} - \theta \\ \mathrm{vech}(\hat{\Sigma} - \Sigma)\end{pmatrix} \overset{d}{\to} N(\mathcal{I}^{-1}\mathcal{J}\mathcal{I}^{-1}),$$

where $\mathcal{J}$ is the asymptotic variance of the score function and $\mathcal{I}$ is (minus) the limit of Hessian matrix for the log-likelihood function.

In practice, we rely on the expression (1) for estimating $\hat{\Sigma}$, and are mainly concerned with computing standard errors for (elements of) $\hat{\theta}$. Fortunately, $\mathcal{I}$ has a block diagonal structure that simplifies the computation of standard errors for θ.

Theorem 2. Given the martingale difference conditions stated in Corollary 1 and Conjecture 1,

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \overset{d}{\to} N\left(0, \hat{\mathcal{I}}_\theta^{-1}\hat{\mathcal{J}}_\theta\hat{\mathcal{I}}_\theta^{-1}\right).$$

In practice, we will use this simplification and compute standard errors for θ using $\hat{\mathcal{I}}_\theta^{-1}\hat{\mathcal{J}}_\theta\hat{\mathcal{I}}_\theta^{-1}$, where $\hat{\mathcal{J}}_\theta$ will be based on the analytical scores derived in Equation (2) and $\hat{\mathcal{I}}_\theta$ will be computed from the numerical Hessian matrix of the log-likelihood function.

3.3 Partial Log-Likelihood for Returns

To have a measure of fit that can be compared with conventional GARCH models, we define

$$\ell_P(r;\theta) = -\frac{1}{2}\sum_{t=1}^{n}\left[\log(2\pi) + \log(h_t) + (r_t - \mu)^2/h_t\right],$$

which is the partial log-likelihood function (for the time series of returns). This quantity is the Kullback–Leibler measure associated with the conditional distribution of returns. So this measure is directly comparable to the log-likelihood obtained from conventional GARCH models, such as the GARCH model and the EGARCH model.

4. EMPIRICAL RESULTS

In this section, we present empirical results using returns and realized measures for 27 liquidly traded stocks and an exchange-traded index fund, SPY, which tracks the S&P 500 index. The series assets coincide with those in Hansen, Huang, and Shek (2012) with the exception that General Motors is omitted from that analysis, due to its 2009 bankruptcy that effectively halted trading of the company for more than a year.

The empirical results illustrate the (rather large) benefits of using realized measures in this framework. For instance, the log-likelihood for returns increases substantially when realized measures are included in the modeling, and γ is found to be significant for all realized measures. Models with multiple realized measures lead to better empirical fits than those with a single realized measure. The best combination of realized measures appears to be the realized variance based on 2 min sampling in conjunction with the daily range. We explore a number of simplifications of the model structure and find $\varphi = 1$ is a useful restrictions that improves the out-of-sample fit. Additional results are presented in the next section where we show that the exponential specification proposed in this article is superior to the original specification by Hansen, Huang, and Shek (2012). Section 5 will also present theoretical insight about the underlying reasons for this.

4.1 Data and Realized Measures

Our full sample spans the period from January 1, 2002, to December 31, 2013. We present results for two out-of-sample periods with either a half or a third of the full sample. We will present extensive out-of-sample results, based on an out-of-sample period that spans the period from January 1, 2008, to December 31, 2013, which is 50% of the full sample. The results we obtained with the shorter out-of-sample period from January 1, 2010, to December 31, 2013 (hence a longer in-sample period) are similar. One of these results were included in the introduction. Our data are an extended version of the data that were analyzed by Hansen, Huang, and Shek (2012). The present dataset is more than 50% longer than that in Hansen, Huang, and Shek (2012), and we include additional realized measures in the present analysis and make use of models with different subsets of eight distinct realized measures. The realized measures were kindly provide to us by Asger Lunde, and these were computed from high-frequency data that were cleaned as detailed in Barndorff-Nielsen et al. (2009).

We remove short trading days from our sample to avoid outliers that would result from the fact that the realized measures, on such days, span a smaller percentage of daily volatility. Specifically, we removed the days where high-frequency data spanned less than 20,000 sec. A typical trading day spans the 6.5 hr (23,400 sec) from 9:30 a.m. to 4:00 p.m. This removes about three daily observations per year, such as the day after Thanksgiving Day and the days around Christmas.

4.1.1 Realized Measures of Volatility. We use eight realized measures in our empirical analysis. These consist of the realized kernel (RK) by Barndorff-Nielsen et al. (2008), the daily range (DR), and six realized variances (RV) that differ in terms of the sampling frequency of intraday returns (ranging from 15 sec intraday returns to 20 min intraday returns).

A relatively simple realized measure of volatility is the sum of squared intraday returns that is known as the realized variance. By dividing some interval of time, $[T_0, T_1]$ say, into n subintervals, $T_0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T_1$, we can define the intraday returns, $r_{i,n} = p_{t_{i,n}} - p_{t_{i-1,n}}$. The realized variance is now defined by $\mathrm{RV}_t^{(n)} = \sum_{i=1}^{n} r_{i,n}^2$, and under ideal circumstances the realized variance is consistent for the quadratic variation. (Note that the quadratic variation is an ex-post measure of volatility, as oppose to h_t that is an ex-ante quantity.) However, it is well known that market microstructure noise becomes increasingly important as $n \to \infty$, which makes the RV an unreliable measure of volatility when n is large, see Zhang, Mykland, and Aït-Sahalia (2005), Bandi and Russell (2008), and Hansen and Lunde (2006).

The RK by Barndorff-Nielsen et al. (2008) is one of several robust measures of volatility, and the variant derived in Barndorff-Nielsen et al. (2011) is robust to general forms of noise. In this article, we adopt the latter variant, which is given by $\mathrm{RK} = \sum_{h=-H}^{H} k(\frac{h}{H+1})\gamma_h$, where $k(x)$ is the Parzen kernel and $\gamma_h = \sum_{i=|h|+1}^{n} r_{i,n} r_{i-h,n}$. The exact computation of this estimator is described in Barndorff-Nielsen et al. (2009).

The daily range is defined by $\mathrm{high}_t - \mathrm{low}_t$ with $\mathrm{high}_t = \max_s p_s$ and $\mathrm{low}_t = \min_s p_s$, where p_t is the logarithmic price and the maximum and minimum are taken over observed prices on the tth trading day. For the sake of convenience, we use the squared daily range,

$$\mathrm{DR}_t = (\mathrm{high}_t - \mathrm{low}_t)^2,$$

because this transforms the range-measure to the same scale as h_t. When log-prices follow a Brownian motion with (constant) variance h_t, then $\log \mathrm{DR}_t \sim N(0.85 + \log h, 0.34)$ whereas $\log(\mathrm{high}_t - \mathrm{low}_t) \sim N(0.43 + \frac{1}{2}\log h, 0.08)$, see Alizadeh, Brandt, and Diebold (2002, Table 1).

4.2 A Comparison of Realized Measure

First, we compare the performance of realized EGARCH models that include a single realized measure and compare the results for the eight realized measures. Later we explore the benefits of using multiple realized measures simultaneously. Results based on open-to-close returns are presented in Table 2, Panel A, and the analogous results for close-to-close results are presented in Panel B.

Interestingly, RV2m often delivers the best out-of-sample fit, albeit in the case of SPY close-to-close returns, the best individual realized measure is the RK. Two realized measures, DR and RV15s, stand out as inferior. In the case of the daily range, this is explained by the DR being the most noisy measure of volatility, which is also evident from it having the largest variance of u_t. In the case of RV15s, its weak out-of-sample performance can be attributed to the RV15s being most sensitive to microstructure noise, because it is sampled at the highest frequency, causing it to be poor signal of the underlying integrated volatility. For open-to-close SPY returns, the RV15s leads to a good in-sample fit for returns. This may be because the SPY is the most actively traded asset in our analysis, making microstructure noise less of an issue. But even in this case, the RV15s falls short in terms of the out-of-sample log-likelihood. Not surprisingly do we find that the estimates for β are close to 1. This is true uniformly across different stocks for all the different realized measures.

There is an inverse relationship between the coefficient of residual measurement error, γ, and its variance σ_u^2 (for individual realized measures, we write σ_u^2 in place of Σ). This is logical because the more accurate is a realized measure, the larger would we expect its coefficient in the GARCH equation to be.

It makes little sense to compare the full log-likelihood $\ell(r, x)$ for different realized measures, because these are log-likelihoods for different time series. So we will focus on the partial likelihood that is a measure of fit for the return data. Figure 1 compares the realized EGARCH model with the conventional EGARCH model in terms of the partial log-likelihood value. The values reported are the average in-sample and out-of-sample improvements in the partial log-likelihood for each of the eight realized measures. The average is taken over the 28 assets. We note that all realized measures lead to better in-sample and out-of-sample performance, which demonstrates the value of incorporating realized measures in the modeling of returns. The realized measures perform similarly in-sample, with the exception of RV15s that yields a smaller gain over the EGARCH model. Out-of-sample the RV2m performs best on average, closely followed by RK and RV5min.

From inspecting the in-sample log-likelihoods, we notice an inverted U-shape across the sampling frequencies for the realized variances. As the sampling frequency increases, the likelihood tends to improve, until a certain point around 2–5 min frequency, after which the average likelihood deteriorates. This

Table 2. Results based on *open-to-close* returns (Panel A) and *close-to-close* returns (Panel B) for eight realized measures of volatility. The left panel presents parameter estimates for SPY returns for four key parameters along with the value of the in-sample and out-of-sample partial log-likelihood function for the case where the sample is split in the middle, January 1, 2008. Each row has the results for one of the eight realized measures in our analysis. The right panel presents the corresponding results for the individual stocks, in the form of average point estimate and average value of the log-likelihood function

	Point estimates and log-likelihoods (SPY)						Averages across all assets					
Panel A: open-to-close returns												
	β	γ	φ	σ_u^2	$\ell_P^{\mathrm{IS}}(r)$	$\ell_P^{\mathrm{OS}}(r)$	β	γ	φ	σ_u^2	$\ell_P^{\mathrm{IS}}(r)$	$\ell_P^{\mathrm{OS}}(r)$
RK	0.968	0.303	1.107	0.122	−1744.51	−1907.90	0.969	0.308	1.116	0.143	−2351.29	−2536.35
DR	0.972	0.146	0.878	0.352	−1775.59	−1951.76	0.978	0.124	1.060	0.346	−2352.03	−2548.08
RV15s	0.976	0.304	1.200	0.107	−1738.41	−1913.24	0.973	0.338	1.091	0.099	−2356.70	−2545.77
RV2m	0.971	0.266	1.117	0.143	−1741.66	−1906.51	0.972	0.291	1.105	0.143	−2351.34	−2532.84
RV5m	0.968	0.242	1.130	0.176	−1745.31	−1907.27	0.970	0.255	1.090	0.185	−2350.39	−2534.40
RV10m	0.968	0.219	1.120	0.211	−1744.76	−1907.32	0.971	0.214	1.081	0.237	−2351.08	−2537.85
RV15m	0.970	0.184	1.160	0.246	−1748.07	−1909.58	0.973	0.184	1.065	0.284	−2351.05	−2539.85
RV20m	0.971	0.186	1.080	0.266	−1744.01	−1906.91	0.974	0.165	1.064	0.321	−2351.89	−2539.43
Panel B: close-to-close returns												
	β	γ	φ	σ_u^2	$\ell_P^{\mathrm{IS}}(r)$	$\ell_P^{\mathrm{OS}}(r)$	β	γ	φ	σ_u^2	$\ell_P^{\mathrm{IS}}(r)$	$\ell_P^{\mathrm{OS}}(r)$
RK	0.971	0.308	1.041	0.108	−1877.40	−2249.72	0.970	0.327	1.044	0.143	−2536.90	−2822.24
DR	0.977	0.104	1.100	0.414	−1877.94	−2278.81	0.980	0.121	1.000	0.428	−2537.98	−2839.90
RV15s	0.978	0.303	1.130	0.096	−1874.24	−2266.12	0.974	0.357	1.021	0.098	−2542.30	−2837.10
RV2m	0.973	0.264	1.052	0.130	−1875.47	−2257.41	0.972	0.308	1.036	0.143	−2537.13	−2819.23
RV5m	0.971	0.229	1.072	0.164	−1879.51	−2259.69	0.971	0.265	1.038	0.187	−2536.70	−2823.33
RV10m	0.972	0.199	1.055	0.201	−1878.13	−2256.50	0.972	0.220	1.027	0.242	−2537.28	−2823.64
RV15m	0.975	0.170	1.024	0.240	−1877.77	−2255.38	0.974	0.191	1.015	0.293	−2537.57	−2829.65
RV20m	0.975	0.158	1.015	0.265	−1877.51	−2253.07	0.975	0.171	1.007	0.333	−2538.37	−2827.62

result is consistent with the literature on market microstructure noise in high-frequency data, which has shown that the mean square error (MSE) of the realized variance, as a function of the sampling frequency, has a U-shape. The U-shape arises because the distortions induced by market microstructure noise increase with the sampling frequency. This distortion eventually dominates the statistical gain from sampling more frequently.

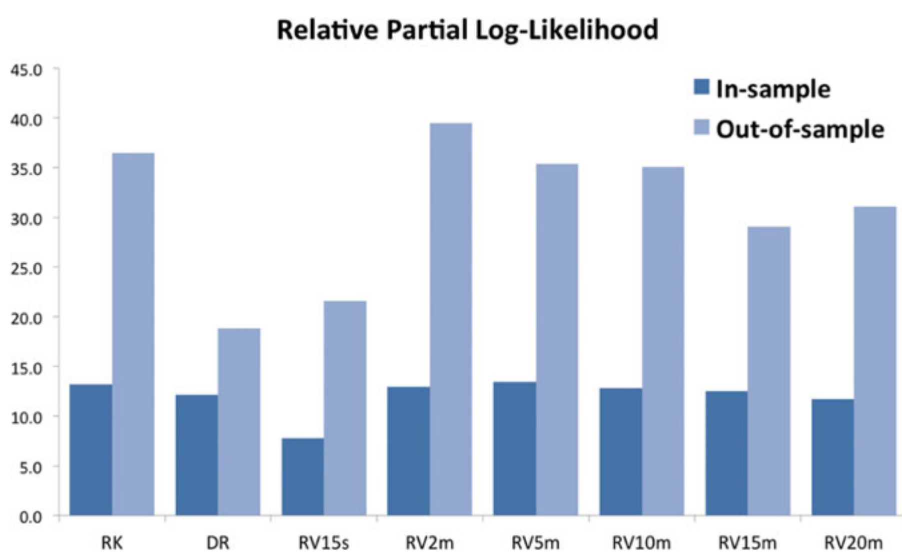

Figure 1. Results for the realized EGARCH model based on each of the eight realized measures. We report the gains in the in-sample and out-of-sample partial log-likelihood relative to that of the conventional EGARCH model. The values reported are the average gain over 28 assets. All realized measures lead to better in-sample and out-of-sample performance. The realized measures perform similarly in-sample, with the exception of RV15s that is somewhat worse on average. Out-of-sample the RV2m performs best on average, closely followed by RK and RV5min.

4.3 Detailed Results for Realized Exponential GARCH With the Realized Kernel

We shall present more detailed results for the realized EGARCH model based on the RK. The following are the results for SPY open-to-close returns for the full sample period:

$$r_t = \underset{(0.013)}{+0.004} + \sqrt{h_t} z_t$$

$$\log h_{t+1} = \underset{(0.003)}{-0.014} + \underset{(0.003)}{0.970} \log h_t - \underset{(0.007)}{0.101} z_t + \underset{(0.004)}{0.044} (z_t^2 - 1) + \underset{(0.020)}{0.375} u_t$$

$$\log x_{\mathrm{RK},t} = \underset{(0.030)}{-0.164} + \underset{(0.029)}{1.086} \log h_t - \underset{(0.007)}{0.092} z_t + \underset{(0.004)}{0.059} (z_t^2 - 1) + u_t,$$

with $\hat{\sigma}_u^2 = \underset{(0.004)}{0.137}$. The numbers in parentheses are the robust standard errors for each of the point estimates.

Estimating the same specification for SPY close-to-close returns yields:

$$r_t = \underset{(0.015)}{+0.030} + \sqrt{h_t} z_t$$

$$\log h_{t+1} = \underset{(0.004)}{-0.005} + \underset{(0.004)}{0.970} \log h_t - \underset{(0.008)}{0.151} z_t + \underset{(0.004)}{0.027} (z_t^2 - 1) + \underset{(0.020)}{0.372} u_t$$

$$\log x_{\mathrm{RK},t} = \underset{(0.026)}{-0.522} + \underset{(0.027)}{0.999} \log h_t - \underset{(0.007)}{0.148} z_t + \underset{(0.004)}{0.025} (z_t^2 - 1) + u_t,$$

with $\hat{\sigma}_u^2 = \underset{(0.003)}{0.131}$.

The estimates are quite similar, with the exception of the intercept parameter in the measurement equation, ξ, which is smaller for close-to-close returns than for open-to-close returns. This is to be expected because it is the same realized kernel estimate that is used in both specification, with the implication that $x_{\mathrm{RK},t}$ is a downward biased measurement of h_t, when the latter is the daily close-to-close volatility. The fact that the coefficient on u_t, γ, is significant shows that the realized measure (RK) provides valuable information about the variation in volatility, over and above that explained by studentized returns, z_t. We also observe that the coefficients for z_t are smaller (more negative) for close-to-close returns, which suggests a higher degree of asymmetry in the leverage effect. The estimate of the mean parameter, μ, is close to zero for open-to-close returns and only slightly positive for close-to-close returns.

The point estimates for each of the assets based on close-to-close returns are presented in Table 3. The analogous results for open-to-close returns are similar and can be found in Table A.1 in Appendix A. The last row of Table 3 has the average estimates across all assets, and it is interesting to observe how similar the point estimates are across assets. Not surprisingly,

Table 3. Estimates for the realized EGARCH model based on the realized kernel (RK) for close-to-close returns. Full sample: January 1, 2002, to December 31, 2013

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	σ_u^2
AA	−0.007	0.044	0.972	0.344	−0.051	0.039	−0.490	1.040	−0.056	0.063	0.141
AIG	−0.030	0.048	0.969	0.541	−0.074	0.045	−0.357	0.897	−0.035	0.052	0.202
AXP	0.050	0.013	0.986	0.390	−0.083	0.041	−0.383	0.990	−0.059	0.065	0.157
BA	0.068	0.022	0.978	0.302	−0.059	0.036	−0.449	1.056	−0.048	0.079	0.144
BAC	0.004	0.026	0.977	0.534	−0.081	0.043	−0.388	0.944	−0.060	0.046	0.167
C	−0.021	0.038	0.971	0.477	−0.084	0.078	−0.341	0.941	−0.061	0.076	0.159
CAT	0.058	0.032	0.973	0.369	−0.056	0.015	−0.584	1.105	−0.064	0.040	0.136
CVX	0.055	0.020	0.966	0.342	−0.078	0.037	−0.345	1.113	−0.104	0.045	0.125
DD	0.037	0.021	0.971	0.412	−0.075	0.022	−0.225	0.968	−0.067	0.046	0.149
DIS	0.054	0.016	0.981	0.326	−0.074	0.023	−0.347	1.006	−0.073	0.050	0.153
GE	0.018	0.013	0.982	0.406	−0.060	0.024	−0.324	0.985	−0.037	0.045	0.160
HD	0.045	0.017	0.980	0.380	−0.058	0.026	−0.248	0.957	−0.041	0.050	0.145
IBM	0.027	0.012	0.973	0.423	−0.073	0.012	−0.374	0.983	−0.061	0.035	0.137
INTC	0.020	0.033	0.973	0.474	−0.050	0.018	−0.281	0.912	−0.034	0.032	0.128
JNJ	0.032	−0.005	0.977	0.344	−0.066	0.038	−0.110	0.970	−0.026	0.062	0.159
JPM	0.018	0.023	0.980	0.440	−0.087	0.057	−0.306	0.946	−0.057	0.063	0.144
KO	0.030	0.001	0.973	0.400	−0.065	0.028	−0.157	0.942	−0.047	0.060	0.150
MCD	0.059	0.006	0.984	0.296	−0.041	0.024	−0.297	1.041	−0.060	0.075	0.167
MMM	0.039	0.014	0.968	0.341	−0.078	0.009	−0.363	1.084	−0.068	0.040	0.153
MRK	0.026	0.022	0.972	0.329	−0.044	0.014	−0.493	1.090	−0.051	0.051	0.181
MSFT	0.031	0.026	0.968	0.449	−0.039	0.011	−0.392	1.000	−0.028	0.030	0.138
PG	0.025	−0.001	0.961	0.365	−0.060	0.024	−0.160	1.088	−0.051	0.056	0.157
T	0.033	0.012	0.975	0.394	−0.058	0.043	−0.153	0.976	−0.061	0.061	0.175
UTX	0.044	0.019	0.968	0.350	−0.101	0.037	−0.272	0.963	−0.055	0.066	0.149
VZ	0.026	0.010	0.979	0.322	−0.061	0.039	−0.192	1.030	−0.055	0.061	0.160
WMT	0.015	0.005	0.979	0.301	−0.035	0.027	−0.235	1.125	−0.027	0.059	0.142
XOM	0.040	0.016	0.967	0.350	−0.087	0.040	−0.285	1.087	−0.108	0.047	0.124
SPY	0.030	−0.005	0.970	0.372	−0.151	0.027	−0.522	0.999	−0.148	0.025	0.131
Average	0.029	0.018	0.974	0.385	−0.069	0.031	−0.324	1.008	−0.059	0.053	0.151

do we find volatility to be highly persistent, which is evident from the estimates of β, that are close to 1 in all cases. Moreover, the tables show that γ is typically estimated to be between 0.35 and 0.45. This parameter may be compared with α in a conventional GARCH model, which measures the coefficient associated with squared returns. The fact that γ is estimated to be several times larger than the typical value for α (about 0.05) reflects the fact that the realized kernel offers a stronger signal about future volatility than does the squared return.

4.4 Simplifying the Structure Through Parameter Restrictions

We seek ways to simplify the model by imposing parameter restrictions that are not at odds with the data. There are several advantages of imposing restrictions. For instance, it can ease the interpretation of the model and can make the estimation of the remaining parameters more efficient.

Imposing the restriction $\mu = 0$ in the mean equation was found to improve the average out-of-sample fit in the shorter sample used in an earlier version of this article. For the longer sample, we use in the present article, the average out-of-sample fit is typically a tad better for the models where μ is unrestricted. The difference can be explained by the longer in-sample period that produces a more accurate estimate of μ. For this reason, we do not impose $\mu = 0$ in our empirical analysis. Nevertheless, we recommend exploring this restriction in practice, in particular when the model is estimated with relatively short sample periods.

Imposing the restriction: $\varphi = 1$ can be motivated by theoretical considerations, specifically that the realized measures are expected to be proportional to h_t. Since we operate with logarithmically transformed quantities, this would naturally lead to $\varphi = 1$. Imposing this constraint makes it easier to interpret certain features of the model, so we examine the validity of this restriction. We estimate the realized EGARCH model with each of the eight realized measures and evaluate the effects on the joint and partial log-likelihood functions by imposing the constraint $\varphi = 1$. The results are presented in Figure 2 where the left panel displays the in-sample and out-of-sample effects on the joint log-likelihood, and the results for the partial log-likelihood are presented in the right panel. The numbers reported in the left panel are $\ell(r, x; \tilde{\theta}, \tilde{\Sigma}) - \ell(r, x, \hat{\theta}, \hat{\Sigma})$ (average over the 28 assets), where $\tilde{\theta}$ and $\tilde{\Sigma}$ are the point estimates from the restricted model (where $\varphi = 1$ is imposed), while $\hat{\theta}$ and $\hat{\Sigma}$ are the point estimates from the unrestricted model. The right panel presents the equivalent statistics for the partial log-likelihood function.

Because the parameters are estimated by maximizing the joint likelihood (in-sample), it is no surprise that the in-sample log-likelihood decreases in value when $\varphi = 1$ is imposed. This is not necessarily the case for the partial log-likelihood and, in fact, the daily range provides an example where the in-sample partial log-likelihood increases by imposing $\varphi = 1$. (The implication is that the marginal in-sample log-likelihood for the realized measures decreases more than the joint log-likelihood). The interesting result in the left panel is that the restriction improves the out-of-sample fit on average. We note that the out-of-sample improvement is about 10 log-likelihood points, for both joint and partial likelihoods. This implies that the gains from imposing $\varphi = 1$ are primarily driven by gains in the partial log-likelihood for returns, and less so by the part of the likelihood that relates to the realized measure. This is obviously desirable if the key objective is a better out-of-sample model for returns. We conclude that $\varphi = 1$ is a reasonable restriction to impose in this framework.

4.5 Realized Exponential GARCH With Multiple Realized Measures

In this section, we present empirical results for realized EGARCH models using multiple realized measures. Models using different combinations of realized measures have been estimated for $M = 2, 3, 4$.

Seven realized EGARCH models are estimated with different pairs of realized measures. We estimate two models with three realized measures, where the labels (RK,DR,RV15s) and (RK,DR,RV2m) identify which realized measures are included in the model. We estimate three models with four realized measures: (RK,RV2m,RV5m,RV20m), (RK,DR,RV5m,RV20m), and (RK,DR,RV15s,RV2m).

The results are reported in Table 4. The first and second columns report the partial log-likelihood for the in-sample and

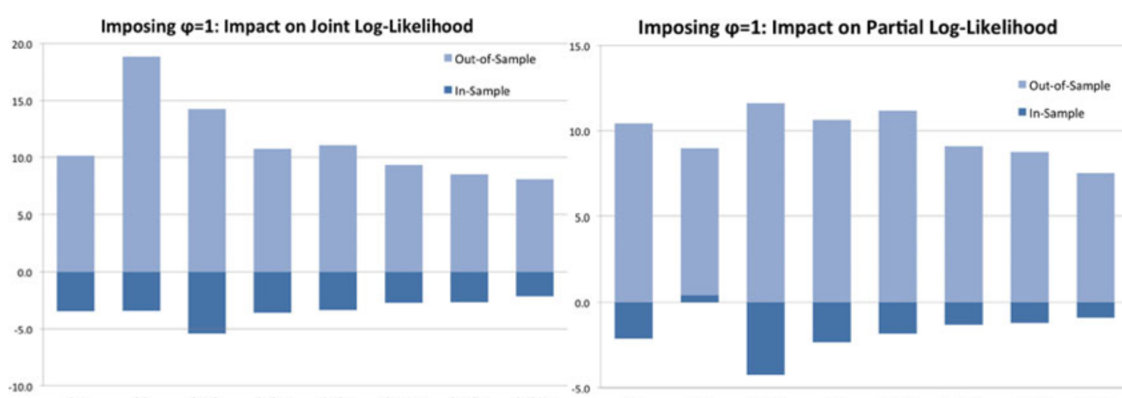

Figure 2. Gains in the joint and partial log-likelihoods by imposing $\varphi = 1$. While restrictions will reduce the value of the in-sample likelihood we note that imposing $\varphi = 1$ leads to improvements in the out-of-sample log-likelihood. Improvements are observed for both the joint and the partial log-likelihoods out-of-sample. The reported values are the average gains across all assets.

Table 4. In-sample and out-of-sample partial log-likelihoods for various specifications with one or more realized measures

	Partial log-likelihood: $\bar{\ell}_r$ (average over assets)		Change in $\bar{\ell}_r$ relative to best specification	
Specification	IS	OS	IS	OS
EGARCH	−2550.10	−2858.71	−13.00	−51.02
RK	−2539.02	−2811.80	−1.92	−4.12
DR	−2537.57	−2831.32	−0.47	−23.64
RV15s	−2546.55	−2825.49	−9.44	−17.80
RV2m	−2539.47	−2808.60	−2.37	−0.91
RV5m	−2538.55	−2812.15	−1.45	−4.46
RV10m	−2538.60	−2814.54	−1.50	−6.86
RV15m	−2538.77	−2820.88	−1.67	−13.19
RV20m	−2539.28	−2820.10	−2.18	−12.41
RK,DR	−2537.98	−2811.28	−0.88	−3.59
RK,RV15s	−2549.53	−2834.44	−12.42	−26.76
RK,RV5m	−2538.52	−2810.83	−1.42	−3.14
RV15s,RV5m	−2546.75	−2827.03	−9.65	−19.35
RV5m,RV20m	−2538.15	−2810.42	−1.04	−2.74
DR,RV5m	−2537.17	−2810.99	−0.07	−3.30
DR,RV2m	−2538.31	**−2807.68**	−1.21	0.00
RK,DR,RV15s	−2548.23	−2834.21	−11.13	−26.52
RK,DR,RV2m	−2537.80	−2809.76	−0.70	−2.08
RK,RV2m,RV5m, RV20m	−2538.48	−2808.60	−1.38	−0.92
RK,DR,RV5m, RV20m	**−2537.10**	−2811.21	0.00	−3.53
RK,DR,RV15s, RV2m	−2547.94	−2833.69	−10.84	−26.01

NOTE: Best performing specification identified by boldface.

out-of-sample periods, respectively. The values reported are the average results for the 28 assets. The last two columns report the values of the log-likelihood relative to that obtained for the specification with the highest average value. In-sample, the largest average partial log-likelihood is (not surprisingly) achieved by a specification with four realized measures (RK, DR, RV5m, and RV20m), whereas the simpler specification with two realized measures, DR and RV2m, has the best out-of-sample fit on average.

From the models with two realized measures, it is interesting to note that including the realized range, DR, tends to improve the out-of-sample likelihood, despite the fact that the DR alone yields the worst out-of-sample performance. This suggests that the daily range contains supplementary information to that provided by the realized kernel and the realized variances. For models that include RV15s we see the exact opposite. When RV15s is included in the specification, the out-of-sample fit deteriorates substantially.

Comparing the out-of-sample partial log-likelihoods with those of the univariate model, we find that incorporating multiple realized measures does improve the value of the likelihood. The results with $M = 3$ realized measures are similar to the case of $M = 2$, with the exception being the case where we include the noise-sensitive realized variance, RV15s.

Several specifications produce very similar in-sample and out-of-sample log-likelihoods. Specifications that include either realized kernel or RV2m in conjunction with the daily range deliver good overall performance, unless the noise-prone realized variance, RV15s, is also included.

4.6 Model Diagnostics

In this section, we inspect some of the model assumptions for the realized EGARCH model estimated with SPY returns and the RK over the full sample.

The primitive assumptions required for the scores to be a martingale difference were identified in Corollary 1. These assumptions require the absence of autocorrelation in z_t, z_t^2, u_t, and u_t^2. Figure 3 presents the first 40 autocorrelations for all four series. The shaded area are the 95% confidence bands based on robust standard errors.

An immediate concern is the fact that the first-order autocorrelations of all four series are significant at the 5% level. While small in absolute values, many coefficients are significant, owing in part to the large sample size with $n = 3005$ daily observations.

For the case of z_t and z_t^2, we do not see more violations than might be attributed to chance. Similar autocorrelations for z_t and z_t^2 were reported in Lunde and Olesen (2013), who applied the realized EGARCH models to energy forward.

For the case of u_t and u_t^2, we see that about 10 of the first 40 autocorrelations are significant at the 5% level, and the assumed constancy of σ_u^2 is challenged by the fact that the first 25 autocorrelations of u_t^2 are positive. If we estimate σ_u^2 year by year from 2002 to 2013, we obtain the following subsample estimates: 0.08, 0.09, 0.10, 0.08, 0.12, 0.18, 0.17, 0.10, 0.16, 0.15, 0.15, and 0.17, which confirm time-variation in σ_u^2. Corollary 1 shows that nonconstancy of σ_u^2 only affects the martingale difference properties of the part of the score that relates to σ_u^2. So the constancy of σ_u^2 is less critical for inference on other parameters. One possible way to model the time variation in σ_u^2 is to include a GARCH structure for u_t. We have explored this extension and found it to have little impact on the estimated parameters, and on average it did not improve the out-of-sample partial log-likelihood for returns, see Appendix C.

Fortunately, the empirical correlations for the pairs (u_t, z_t) and (u_t, z_t^2) are numerically equal to zero. We suspect that this is a feature of the scores derived in Theorem 1, albeit it is not immediately clear from the expressions.

Next we turn to the Gaussian assumptions used in the likelihood estimation. Lunde and Olesen (2013) reported large deviation from Gaussianity in their $\hat{z}_t$, that are deduced from energy forward rates. Here, we find the Gaussian assumption to be less at odds with our returns, see Figure 4. In the case of u_t, the empirical distribution has fatter tails than that of the normal distribution, which can be attributed to the assumed constancy of σ_u^2, which appears to be at odds with the data. In addition, outliers in the realized measures, and/or outliers in z_t that affect u_t through the leverage function $\delta(z)$ can result in outliers for u_t.

Overall, it appears that the realized EGARCH model with a Gaussian specification, does a reasonably good job at modeling the returns, but falls short in terms of properly describing the

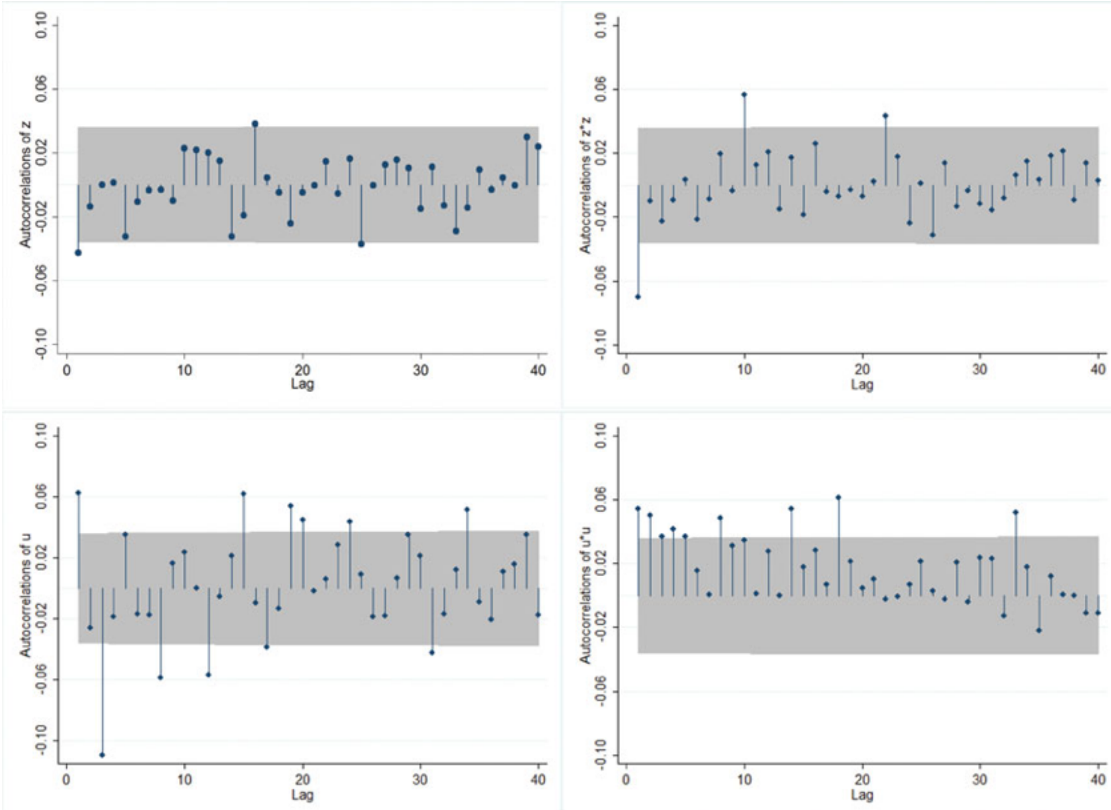

Figure 3. Autocorrelations (ACFs) in residuals and squared residuals along with 95% confidence bands, which are illustrated by the shaded area. The ACFs and their robust standard errors were computed with Stata using the Bartlett command.

dynamic properties of the realized kernel. Primarily, because of unmodeled time variation in σ_u^2.

5. RELATION TO EARLIER SPECIFICATION IN HANSEN, HUANG, AND SHEK (2012)

In this section, we compare the realized exponential GARCH specification with the earlier specification in Hansen, Huang, and Shek (2012). The latter was formulated for a single realized measure, so we focus on this special case in our comparison. The logarithmic specification in Hansen, Huang, and Shek (2012) takes the form:

$$\log h_t = \tilde{\omega} + \tilde{\beta} \log h_{t-1} + \gamma \log x_{t-1},$$
$$\log x_t = \xi + \varphi \log h_t + \delta(z_t) + u_t.$$

Compared to the realized EGARCH specification, we note that the GARCH equation has the realized measure, x_{t-1}, instead of

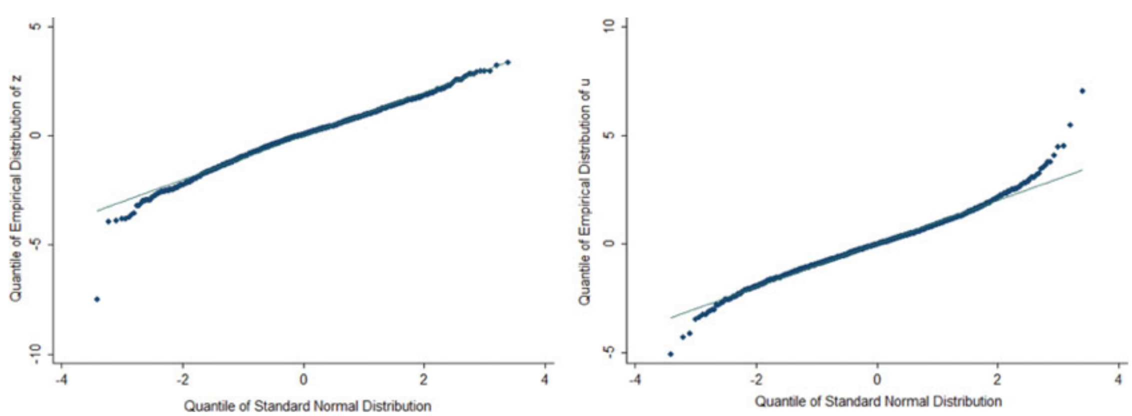

Figure 4. QQ plots for $\hat{z}_t$ and $\hat{u}_t$. The Gaussian specification is fairly reasonable for the case of z_t, albeit a higher frequency of negative returns is observed in the data, including an extreme outlier that occurred on February 27, 2007. This outlier has been associated a computer glitch on the New York Stock Exchange on that day.

u_{t-1}, and lacks a leverage function, $\tau(z_{t-1})$. By substitution we have

$$\log h_t = \omega + \beta h_{t-1} + \gamma\delta(z_{t-1}) + \gamma u_{t-1}, \qquad (4)$$

where $\omega = \tilde{\omega} + \gamma\xi$ and $\beta = \tilde{\beta} + \gamma\varphi$. It is now evident that this model is nested in the realized EGARCH model, as the original specification arises by imposing two restrictions: (i) Proportionality of the two leverage functions, $\tau(z) = \gamma\delta(z)$, and (ii) that their relative magnitude is exactly γ (the coefficient for u_{t-1}). Our empirical analysis will highlight the benefits of relaxing these constraints, and we will first provide some theoretical insight about this below.

5.1 Interpreting the Generalized Structure

Before we present the empirical comparison of the two specifications, we will motivate the need for the more flexible structure of the realized EGARCH specification.

The realized measures, x_t, are estimates of the quadratic variation, which we can denote by y_t. For instance, the realized kernel is consistent for y_t as the number of intraday returns, n, increases. In fact, the variant of the realized kernel used in this article is such that $x_t - y_t = O_p(n^{-1/5})$ under suitable conditions, see Barndorff-Nielsen et al. (2011). More generally, we can view each of the realized measures as noisy measure of y_t with varying degrees of accuracy. This translates into $\log x_t$ being a noisy measure of $\log y_t$, with a bias that is influenced by the sampling error of the realized measure. So we introduce $\eta_t = \log x_t - \log y_t$ and $\zeta_t = \log y_t - \log h_t$ and label these as *estimation error* and *volatility shock*, respectively. While it is plausible that the volatility shock influences the dynamics of volatility, there is little reason to expect that the estimation error has any impact on future volatility. This follows from the fact that η_t is specific to the realized measure and simply reflects our inability to perfectly estimate y_t from a finite number of observations.

From the measurement equation, assuming $\varphi = 1$, we have

$$\xi + \delta(z_t) + u_t = \eta_t + \zeta_t.$$

This enables us to interpret ξ and relate $\delta(z_t)$ and u_t to the estimation error and the volatility shock. First, we note that ξ is tied to the sampling error of the realized measure. For a realized measure that is unbiased for y_t, it follows by Jensen's inequality that a larger sampling error decreases $\mathrm{E}\eta_t$. Thus, if we compare two unbiased realized measures, we should expect the more accurate one to have the larger (less negative) value of ξ. Second, since η_t is tied to sampling error that (in the limit for some of the realized measures) is independent of the observed processes, it follows that the leverage function, $\tau(z_t)$, is linked to the volatility shock ζ_t, albeit there will be residual randomness in ζ_t that cannot be explained by the studentized return, z_t, alone. Consequently, the residual measurement shock u_t will be a mixture of the estimation error, η_t, and the residual randomness $\zeta_t - \delta(z_t)$. For this reason, we should expect $\delta(z_t)$ to be more important in describing the dynamic variation in volatility than u_t. A limitation of the original realized GARCH specification is that it implicitly imposes $\delta(z_t)$ and u_t to have the same coefficient in the GARCH equation, see (4).

5.2 Empirical Comparisons to Realized GARCH

Before comparing the realized GARCH and realized EGARCH models, we estimate a hybrid model where we relax the constraint that $\delta(z_t)$ and u_t have the same coefficient in the GARCH equation. This is achieved by imposing $\tau(z) = \kappa\delta(z)$ in the GARCH equation of the EGARCH model, where κ is a free parameter. The realized GARCH model corresponds to the case $\kappa = \gamma$. Figure 5 presents average estimates of γ and κ for the various realized measures, where the averages were taken across assets. Across stocks we typically find the value of γ to be much smaller than κ This can be seen in the left panel, where γ is typically estimated to be about 50% smaller than κ. This observation is consistent with our interpretations of $\delta(z_t)$ and u_t and their relations to estimation error and volatility shocks. Detailed results for the individual stocks are presented in the appendix. The right panel presents more detailed results for the ratio of γ to κ, using standard boxplots. The boxes identifies the first, second, and third quantiles and the whiskers reveal the dispersion to the minimum and

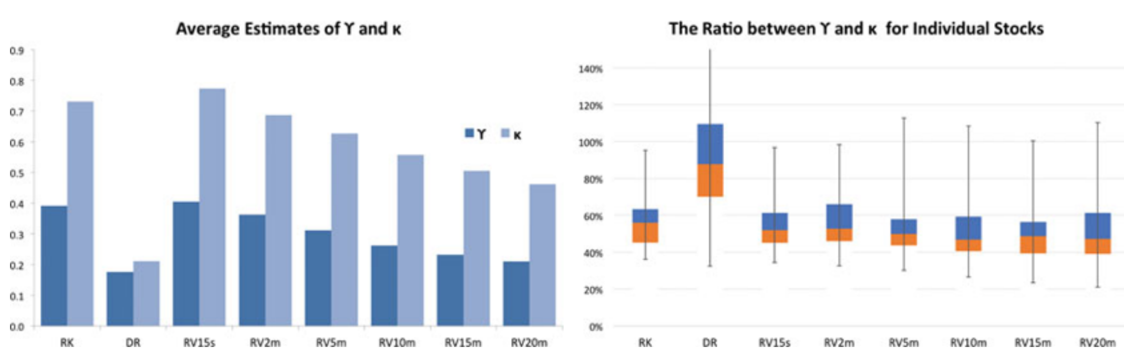

Figure 5. Averaged estimates for γ and κ in the left panel. Right panel presents boxplots of the ratio $\hat{\gamma}/\hat{\kappa}$ using the 28 estimates, one ratio for each of the assets.

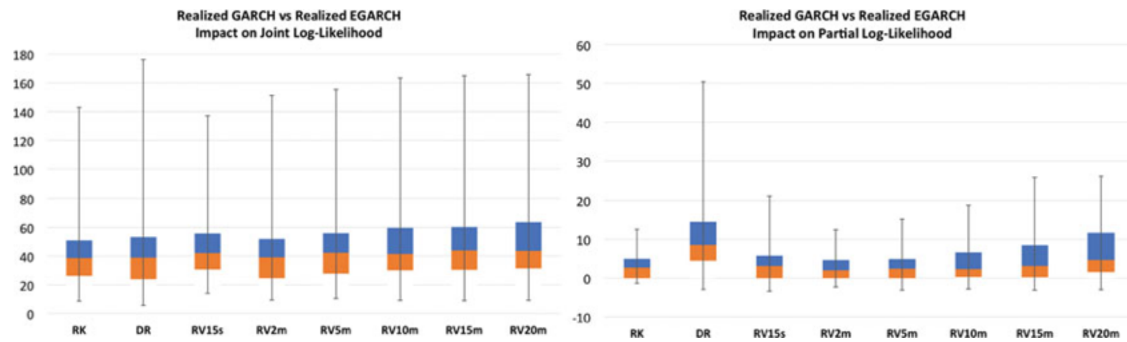

Figure 6. Average impact on the joint and partial log-likelihoods from realized GARCH to realized EGARCH.

maximum value of the ratio, across the 28 assets in our empirical analysis.

Naturally, the realized EGARCH model is more flexible than merely allowing $\gamma \neq \kappa$ and Figure 6 presents likelihood ratio statistics from the comparison of the realized EGARCH model and the nested realized GARCH model over the full sample. Results for the joint likelihood are in the left panel and the results for the partial log-likelihood are in the right panel. We observe substantial gains in the joint likelihood, gains that exceed what can be attributed to chance. Recall that the realized EGARCH model has just two additional parameters. The improvements are also seen for the partial log-likelihood in most cases, but the improvements are more modest. This shows that much of the observed improvements in the joint likelihood can be attributed to improved model fit of the realized measures.

Additional insight about the value of the realized EGARCH structure is evident from the news impact curve. This curve was introduced by Engle and Ng (1993), and is used to illustrate the impact that return shocks has on volatility. Figure 7 plots the impact that z_t has on h_{t+1} measured in percentages, as defined by $\mathrm{E}(\log h_{t+1}|z_t = z) - \mathrm{E}(\log h_{t+1})$. This news impact curve is simply given by $\tau(z)$, in the case of the realized EGARCH model and the two variants of the EGARCH model, and $\gamma\delta(z)$ in the case of the realized GARCH model.

As is evident from Figure 7, the generalized structure of the realized EGARCH model has profound effect on the news impact curve, and results in a curve that is far more similar to that of the EGARCH models.

The realized exponential GARCH model has been applied in several empirical studies, since the model was proposed in the first version of the present article. An extensive comparison of the realized GARCH model, in the context of energy forward prices, is undertaking in Lunde and Olesen (2013). They detailed how multi-period forecasting can be conducted, either by simulating $z_t \sim \mathrm{iid}N(0, 1)$ and $u_t \sim \mathrm{iid}N(0, \hat{\Sigma}_u)$, or by bootstrapping the residual $(\hat{z}_t, \hat{u}_t')'$, see Lunde and Olesen (2013, sec. 6). Their empirical results strongly support the realized EGARCH model, in particular, at the longer horizons, such as 1 week ahead and 1 month

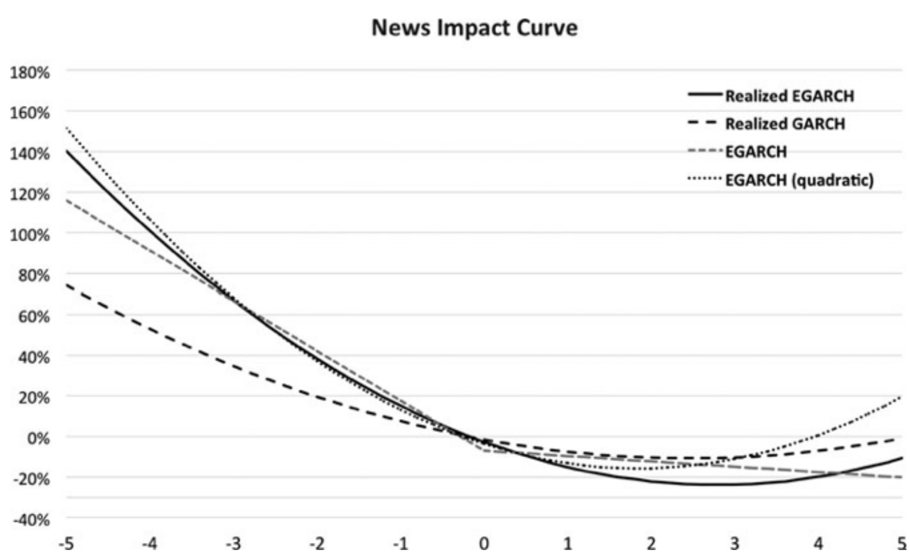

Figure 7. News impact curve for the realized EGARCH, realized GARCH, and the two variants of the conventional EGARCH model, which were defined in the introduction.

ahead forecasting, for a model that is estimated with daily data.

6. CONCLUSION

We have introduced a new and improved variant of the realized GARCH model, which is characterized by two innovations. First, the new model includes an explicit leverage term in the GARCH equation and second, the model enables the inclusion of multiple realized measures in the modeling. The advantages of the new structure were documented in the form of better empirical fit in the time series we have analyzed. In our empirical analysis, we also explored some simplifications and concluded that $\varphi = 1$ is coherent with the data. This is useful because it simplifies the interpretation of the model and facilitates a simpler and more accurate estimation of the model.

We included and compared eight realized measures of volatility in the analysis, a realized kernel (RK), the daily range (DR), and six realized variances, which were computed for various sampling frequencies of intraday returns, ranging from 15 sec returns to 20 min returns. Any of these realized measures contain useful information for the modeling of volatility. The daily range adds the least in terms of empirical fit, and the realized variance based on 2 min sampling adds the most on average. For the realized variances, we find, not surprisingly, that the performance improves as the sampling frequency increases, except for the highest sampling frequency where market microstructure noise becomes a dominating factor. The realized variance based on 15 sec intraday returns, RV15s, can in some cases improve the in-sample fit, even though the effects of market microstructure noise are evident from the point estimates. Out-of-sample, however, the RV15s typically results in a disappointing fit that is barely better than that of the daily range. The core of the problems is that the dynamic properties of RV15s are influenced by features of the market microstructure noise, which causes the RV15s to "hijack" the latent volatility variable, h_t, to (partly) track these noise features instead of the intended purpose, which is to track the conditional volatility of returns.

The extension to multiple realized measures of volatility was found to be beneficial. Not only does multiple realized measure lead to substantial improvements of the empirical fit in-sample, it also improves the out-of-sample fit. The latter implies that the population benefits from adding multiple measures outweighs the drawback from having to estimate additional parameters in these models. In terms of the average out-of-sample fit of the log-likelihood for returns, the best combination of realized measures was one with two realized measures—the daily range paired with the realized variance based on 2 min sampling. While the daily range, in isolation, yields the smallest empirical gains of all realized measures, it does contain valuable information that is orthogonal to that of other realized measures. Given the construction of the realized measures, it is perhaps, not surprising that the daily range, albeit relatively noisy, does capture information that is distinct from that contained in the other realized measures.

Realized measures have proven to be very valuable in GARCH modeling. When estimating a standard GARCH model, the lagged squared returns are typically estimated to have a coefficient around 5%, which causes GARCH models to be slow at adjusting the level of volatility. Put simply, it takes many consecutive large squared returns for a GARCH model to realize that volatility has jumped to a new higher level. Including a realized measure in the GARCH equation will typically lead to its estimated coefficient to be estimated to about 35%–45%, and the inclusion of the realized measures will often cause the squared return to be insignificant. A larger coefficient associated with the realized measure makes the model far more adaptive to sudden changes in volatility, which has obvious benefits. For instance, a realized GARCH model fares far better during the financial crises than does a conventional GARCH model. Despite these benefits, it is important to be aware of a potential drawback of the coefficient, γ, being relatively large. The larger coefficient in the GARCH equation implies that an outlier in the realized measure will cause more havoc to h_t. This problem is less pronounced in conventional GARCH models because α is small. The key message we want to make is the following: A larger coefficient in the GARCH equation requires a higher degree of responsibility in terms of including well-behaved realized measures of volatility. In our empirical analysis, we found that additional data monitoring is required once realized measures are included in the model. For instance, we chose to exclude "short" trading days (mostly days around Christmas and Thanksgiving) from the analysis, because the realized measures from such days are exceptionally small. Not only because the period with high-frequency data is shorter, but also because such days tend to be quiet day with relatively low levels of volatility. Conventional GARCH models do not require the same degree of careful monitoring, because an outlier in returns has a smaller impact on the model-implied volatility.

Finally, the model proposed in this article included a single lag of h_t and u_t in the GARCH equation. This framework is easy to extend to include multiple lags of h_t and u_t, say p and q lags, respectively. It would be natural to label this model as the RealEGARCH(p, q) model, and the likelihood analysis in Section 3 can be adapted to cover this case at the expense of a more complex exposition.

The specification analysis in Section 4.6 revealed evidence of heteroscedasticity in measurement equation's innovations. While this heteroscedasticity is not critical for the asymptotic analysis, aside from the inference that concerns σ_u^2, one could consider the generalized model structure to accommodate time variation in σ_u^2. We have explored ARCH and GARCH structure for u_t, and did not find such to affect other parameter estimates, nor did it improve the partial log-likelihood for returns.

APPENDIX A: APPENDIX OF PROOFS

Proof of Lemma 1. From $z_t = (r_t - \mu)e^{-\tilde{h}_t/2}$, we have that

$$\dot{z}_t = \partial z_t/\partial \tilde{h}_t = -\frac{1}{2}z_t. \tag{A.1}$$

Similarly for $u_{k,t} = \tilde{x}_t - \varphi_k\tilde{h}_t - \delta_k' b(z_t)$, we find that $\dot{u}_{k,t} = \partial u_{k,t}/\partial \tilde{h}_t = -\varphi_k - \delta_k'\dot{b}_t\dot{z}_t = \frac{1}{2}\delta_k'\dot{b}_t z_t - \varphi_k$, so that $\dot{u}_t = \partial u_t/\partial \tilde{h}_t$ (the column vector whose kth element is $\dot{u}_{k,t}$) can be expressed as

$$\dot{u}_t = \frac{1}{2}D\dot{b}_t z_t - \varphi. \tag{A.2}$$

Now recall, $\tilde{h}_{t+1} = g_t'\lambda$, where $g_t' = (1, \tilde{h}_t, a_t', u_t')$. Thus the object we seek is given by

$$(\partial g_t'/\partial \tilde{h}_t)\lambda = \big(0, 1, \dot{z}_t\dot{a}_t', \dot{u}_t'\big)\,\lambda = \beta + \dot{z}_t\dot{a}_t'\tau + \left(\frac{1}{2}D\dot{b}_t z_t - \varphi\right)'\gamma = A(z_t).$$

For the second result, $-2\log\ell_t/\partial\tilde{h}_t = [\tilde{h}_t + z_t^2 + K\log(2\pi) + \log|\Sigma| + u_t'\Sigma^{-1}u_t]/\partial\tilde{h}_t$, we note that $\partial z_t^2/\partial\tilde{h}_t = -z_t^2$ using (A.1), and the result now follows by combining $\partial u_t'\Sigma^{-1}u_t/\partial u_t' = 2u_t'\Sigma^{-1}$ and (A.2). □

Proof of Lemma 2. First we note that

$$z_t = (r_t - \mu)e^{-\tilde{h}_t/2} \qquad \text{and} \qquad u_{k,t} = \tilde{x}_t - \varphi_k\tilde{h}_t - \delta_k' b(z_t),$$

only depend on λ through $\tilde{h}_t$ (the latter directly and indirectly via z_t). Consequently, $g_t' = (1, \tilde{h}_t, a_t', u_t')$ only depends on λ through $\tilde{h}_t$ so that $\partial\tilde{h}_{t+1}/\partial\lambda =$

$$\frac{\partial\tilde{h}_{t+1}}{\partial\lambda} = \lambda'\frac{\partial g_t}{\partial\tilde{h}_t}\frac{\partial\tilde{h}_t}{\partial\lambda} + g_t = A(z_t)\dot{h}_{\lambda,t} + g_t.$$

The second result is derived similarly, albeit with some additional terms because z_t (and hence u_t) depends on μ though another channel than $\tilde{h}_t$. Specifically,

$$\dot{z}_{\mu,t} = \frac{\partial z_t}{\partial\mu} = -\frac{1}{2}z_t\dot{h}_{\mu,t} - h_t^{-\frac{1}{2}},$$

which has implications for $\dot{u}_{\mu,t} = \partial u_t/\partial\mu$ so that

$$\dot{u}_{\mu,t} = -\varphi\dot{h}_{\mu,t} - D\dot{b}_t\dot{z}_{\mu,t} = \left(-\varphi + \frac{1}{2}D\dot{b}_t z_t\right)\dot{h}_{\mu,t} + D\dot{b}_t h_t^{-\frac{1}{2}}$$

so that

$$\begin{aligned}\dot{h}_{\mu,t+1} &= \beta\dot{h}_{\mu,t} + \gamma'\dot{u}_{\mu,t} + \tau'\dot{a}_t\dot{z}_{\mu,t}\\ &= \left[\beta - \gamma'\varphi + \frac{1}{2}(\gamma' D\dot{b}_t - \tau'\dot{a}_t)z_t\right]\dot{h}_{\mu,t} + (\gamma' D\dot{b}_t - \tau'\dot{a}_t)h_t^{-\frac{1}{2}}\end{aligned}$$

and the result follows. □

Proof of Theorem 1. From Lemma 1, we have $\frac{\partial\tilde{h}_{t+j}}{\partial\tilde{h}_t} = \prod_{i=0}^{j-1}A(z_{t+i})$, and the first result follows by the chain rule. For μ, we note that μ influences ℓ_t through $\tilde{h}_t$ and through its direct impact on z_t, which is the second term of $\dot{z}_{\mu,t} = \partial z_t/\partial\mu = -\frac{1}{2}z_t\dot{h}_{\mu,t} - h_t^{-\frac{1}{2}}$ so that

$$\begin{aligned}-2\frac{\partial\ell_t}{\partial\mu} &= B(z_t, u_t)\dot{h}_{\mu,t} + (-2\partial\ell_t/\partial z_t)\left(-h_t^{-\frac{1}{2}}\right)\\ &= B(z_t, u_t)\dot{h}_{\mu,t} + (2z_t + 2u_t'\Sigma^{-1}\partial u_t/\partial z_t)\left(-h_t^{-\frac{1}{2}}\right)\\ &= B(z_t, u_t)\dot{h}_{\mu,t} + 2\big[z_t + u_t'\Sigma^{-1}\big(-D\dot{b}_{z_t}\big)\big]h_t^{-\frac{1}{2}},\end{aligned}$$

which establishes the second element of $\frac{\partial\ell_t}{\partial\theta}$. Next, λ only impacts ℓ_t through $\tilde{h}_t$ so the third element of $\frac{\partial\ell_t}{\partial\theta}$ follows by combining the results in Lemmas 1 and 2. Next consider the derivatives with respect to ψ_k, which only affects ℓ_t through $u_{k,t}$. Recall $-2\partial\ell_t/\partial u_t = \partial(u_t'\Sigma^{-1}u_t)/\partial u_t = 2\Sigma^{-1}u_t$ so that $\partial(u_t'\Sigma^{-1}u_t)/\partial u_{k,t} = 2e_k'\Sigma^{-1}u_t$, where e_k is the kth unit vector. Since $\partial u_{k,t}/\partial\psi_k = -m_t$, we have

$$-2\partial\ell_t/\partial\psi_k = 2e_k'\Sigma^{-1}u_t(-m_t)] = -2\big(e_k'\Sigma^{-1}u_t\big)m_t.$$

By using the fact that for vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, we have that

$$\begin{pmatrix} e_1'ab\\ \vdots\\ e_k'ab\\ \vdots\\ e_n'ab\end{pmatrix} = \begin{pmatrix} a_1 b\\ \vdots\\ a_k b\\ \vdots\\ a_n b\end{pmatrix} = a\otimes b.$$

It follows that $-2\partial\ell_t/\partial(\psi_1', \ldots, \psi_K')' = (\Sigma^{-1}u_t)\otimes m_t$. □

Proof of Theorem 2. We first show the information matrix $\mathcal{I}$ is block diagonal. For each $i, j, k = 1, \ldots, K$, we have

$$\begin{aligned}\frac{\partial^2\ell_t}{\partial(\Sigma^{-1})_{ij}\partial\lambda} &= -\frac{1}{2}(u_{i,t}\dot{u}_{j,t} + u_{j,t}\dot{u}_{i,t})\dot{h}_t\\ \frac{\partial^2\ell_t}{\partial(\Sigma^{-1})_{ij}\partial\psi_k} &= -\frac{1}{2}(u_{i,t}m_{k,j,t} + u_{j,t}m_{k,i,t}),\end{aligned}$$

where $m_{k,i,t} = \frac{\partial u_{i,t}}{\partial\psi_k} = m_{k,t}$ when $k = i$ and 0 otherwise. Hence, we have

$$\begin{aligned}\mathrm{E}\left[\frac{\partial^2\ell(r, x; \theta, \Sigma^{-1})}{\partial(\Sigma^{-1})_{ij}\partial\lambda}\right] &= \mathrm{E}\left[\sum_{t=1}^{n} -\frac{1}{2}(u_{i,t}\dot{u}_{j,t} + u_{j,t}\dot{u}_{i,t})\dot{h}_t\right] = 0,\\ \mathrm{E}\left[\frac{\partial^2\ell(r, x; \theta, \Sigma^{-1})}{\partial(\Sigma^{-1})_{ij}\partial\psi_k}\right] &= \mathrm{E}\left[\sum_{t=1}^{n} -\frac{1}{2}(u_{i,t}m_{k,j,t} + u_{j,t}m_{k,i,t})\right] = 0.\end{aligned}$$

Since all cross terms are zero, the information matrix $\mathcal{I}$ is block triangular, so that

$$\begin{aligned}\mathrm{avar}(\hat{\theta}, \mathrm{vech}\hat{\Sigma}) &= \begin{pmatrix}\mathcal{I}_\theta^{-1} & 0\\ 0 & \mathcal{I}_\Sigma^{-1}\end{pmatrix}\begin{pmatrix}\mathcal{J}_\theta & \mathcal{J}_{\theta\Sigma}\\ \mathcal{J}_{\Sigma\theta} & \mathcal{J}_\Sigma\end{pmatrix}\begin{pmatrix}\mathcal{I}_\theta^{-1} & 0\\ 0 & \mathcal{I}_\Sigma^{-1}\end{pmatrix}\\ &= \begin{pmatrix}\mathcal{I}_\theta^{-1}\mathcal{J}_\theta\mathcal{I}_\theta^{-1} & \mathcal{I}_\theta^{-1}\mathcal{J}_{\theta\Sigma}\mathcal{I}_\Sigma^{-1}\\ \mathcal{I}_\Sigma^{-1}\mathcal{J}_{\Sigma\theta}\mathcal{I}_\theta^{-1} & \mathcal{I}_\Sigma^{-1}\mathcal{J}_\Sigma\mathcal{I}_\Sigma^{-1}\end{pmatrix}.\end{aligned}$$

We can see the asymptotic covariance matrix for $\hat{\theta}$ is just $\mathcal{I}_\theta^{-1}\mathcal{J}_\theta\mathcal{I}_\theta^{-1}$. □

APPENDIX B: ADDITIONAL EMPIRICAL RESULTS

Table B.1. Estimates for the realized EGARCH model based on the realized kernel (RK) for open-to-close returns. Full sample: January 1, 2002, to December 31, 2013

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	σ_u^2
AA	−0.113	0.037	0.969	0.324	−0.049	0.049	−0.183	1.094	−0.040	0.075	0.140
AIG	−0.075	0.040	0.967	0.570	−0.078	0.042	−0.098	0.889	−0.027	0.045	0.209
AXP	0.043	0.009	0.986	0.378	−0.064	0.050	−0.147	1.037	−0.033	0.079	0.156
BA	0.020	0.014	0.979	0.283	−0.041	0.034	−0.161	1.150	−0.027	0.079	0.148
BAC	−0.002	0.018	0.975	0.478	−0.074	0.078	−0.100	0.953	−0.048	0.088	0.155
C	−0.077	0.029	0.968	0.463	−0.070	0.082	−0.100	0.991	−0.044	0.095	0.157
CAT	−0.001	0.022	0.972	0.338	−0.050	0.050	−0.140	1.066	−0.042	0.073	0.133
CVX	0.020	0.008	0.968	0.363	−0.063	0.045	0.005	1.109	−0.076	0.061	0.129
DD	0.001	0.012	0.973	0.360	−0.055	0.037	−0.011	1.045	−0.048	0.068	0.147
DIS	0.063	0.010	0.981	0.310	−0.057	0.033	−0.057	1.104	−0.044	0.069	0.153
GE	−0.018	0.007	0.979	0.350	−0.043	0.050	−0.018	1.044	−0.019	0.076	0.153
HD	0.031	0.013	0.978	0.334	−0.047	0.038	−0.090	1.076	−0.039	0.073	0.140
IBM	0.087	0.001	0.974	0.397	−0.058	0.038	0.037	0.989	−0.035	0.054	0.140
INTC	−0.016	0.022	0.972	0.407	−0.045	0.051	0.027	0.985	−0.023	0.070	0.122
JNJ	0.025	−0.011	0.975	0.350	−0.045	0.033	0.159	1.046	−0.002	0.062	0.163
JPM	0.000	0.018	0.978	0.430	−0.066	0.061	−0.072	0.999	−0.042	0.080	0.142
KO	0.033	−0.006	0.972	0.367	−0.037	0.038	0.086	1.020	−0.025	0.073	0.152
MCD	0.041	0.002	0.986	0.288	−0.029	0.030	−0.001	1.014	−0.040	0.078	0.173
MMM	0.027	0.003	0.967	0.345	−0.056	0.021	0.011	1.092	−0.043	0.051	0.158
MRK	0.008	0.012	0.974	0.304	−0.031	0.017	−0.145	1.151	−0.030	0.054	0.188
MSFT	0.033	0.010	0.971	0.404	−0.045	0.052	0.080	0.948	−0.043	0.075	0.131
PG	0.068	−0.011	0.962	0.376	−0.047	0.027	0.117	1.052	−0.027	0.053	0.164
T	−0.019	0.005	0.977	0.376	−0.045	0.041	0.092	0.985	−0.047	0.063	0.178
UTX	0.003	0.009	0.967	0.344	−0.074	0.039	0.009	1.025	−0.033	0.068	0.152
VZ	−0.019	0.005	0.978	0.343	−0.043	0.041	0.054	1.040	−0.036	0.067	0.162
WMT	0.015	−0.001	0.978	0.312	−0.027	0.036	0.082	1.081	−0.009	0.068	0.145
XOM	0.051	0.005	0.968	0.363	−0.062	0.046	0.042	1.103	−0.072	0.061	0.128
SPY	0.004	−0.014	0.969	0.375	−0.101	0.044	−0.164	1.086	−0.093	0.059	0.138
Average	0.008	0.010	0.974	0.369	−0.054	0.043	−0.024	1.042	−0.039	0.068	0.152

APPENDIX C: GENERALIZATION WITH ARCH AND GARCH STRUCTURES IN U_T

The section with diagnostic analysis suggested the possibility of time variation in the volatility of u_t. In this section, we extend the realized EGARCH model to accommodate ARCH/GARCH effects in u_t. This requires us to introduce an additional latent variable, h_t^u, which denotes the conditional volatility of u_t. With a single realized measure, we consider the following two GARCH equations for the innovations in the measurement equation:

- Realized EGARCH with ARCH in u_t
$$h_t^u = \omega^u + \alpha^u u_{t-1}^2$$
- Realized EGARCH with GARCH in u_t
$$h_t^u = \omega^u + \alpha^u u_{t-1}^2 + \beta^u h_{t-1}^u.$$

It is straightforward to modify the likelihood function by replacing the constant variance σ_u^2 with time varying h_t^u. While we no longer have closed-form expressions for the score functions, it is still a simple matter to obtain the maximum likelihood estimator by simple maximization.

We present the estimated parameters for close-close returns of SPY over the full-sample for the case where with realized kernel is used the realized measure.

From the estimated models and the point estimates reported in Tables C.1 and C.2 and the impact these extensions have on the log-likelihoods, reported in Table C.3, we make the following observations:

- ARCH and GARCH effect are significant at the 5% level. This is evident from the joint log-likelihoods and (unreported) t-statistics for α^u and β^u.
- The volatility of u_t, measured by the sum of $\alpha^u + \beta^u$, on average, is less persistent than the return volatility.
- Including the ARCH or GARCH effect in u_t does not substantially affect the estimates of other parameters in the models.
- The partial log-likelihood for returns is, on average, not improved by adopting an ARCH or GARCH structure for u_t.

Thus, for the purpose of modeling returns, there are no apparent gains from adopting an ARCH/GARCH structure for u_t.

Table C.1. Estimates for the realized EGARCH model with ARCH effect in u_t based on the realized kernel (RK) for close-to-close returns. Full sample: January 1, 2002, to December 31, 2013

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	w_u^2	α^u
AA	−0.008	0.046	0.970	0.328	−0.052	0.042	−0.501	1.045	−0.056	0.064	0.130	0.077
AIG	−0.031	0.048	0.966	0.525	−0.088	0.047	−0.329	0.874	−0.035	0.048	0.167	0.184
AXP	0.050	0.013	0.986	0.390	−0.085	0.042	−0.382	0.990	−0.060	0.064	0.150	0.040
BA	0.069	0.022	0.978	0.302	−0.059	0.036	−0.448	1.055	−0.047	0.079	0.142	0.009
BAC	0.000	0.028	0.975	0.499	−0.088	0.056	−0.394	0.948	−0.062	0.046	0.156	0.058
C	−0.021	0.037	0.971	0.478	−0.086	0.079	−0.337	0.937	−0.061	0.075	0.152	0.047
CAT	0.058	0.033	0.972	0.360	−0.057	0.017	−0.590	1.109	−0.064	0.039	0.128	0.058
CVX	0.056	0.019	0.967	0.328	−0.078	0.038	−0.342	1.110	−0.105	0.047	0.114	0.085
DD	0.035	0.021	0.972	0.373	−0.073	0.030	−0.222	0.963	−0.067	0.049	0.131	0.128
DIS	0.056	0.015	0.982	0.318	−0.073	0.022	−0.345	1.002	−0.071	0.048	0.141	0.079
GE	0.016	0.011	0.984	0.361	−0.065	0.034	−0.340	0.999	−0.039	0.047	0.141	0.116
HD	0.044	0.017	0.980	0.369	−0.061	0.030	−0.249	0.958	−0.042	0.050	0.132	0.087
IBM	0.027	0.011	0.975	0.409	−0.073	0.013	−0.375	0.983	−0.061	0.036	0.130	0.053
INTC	0.020	0.030	0.975	0.458	−0.050	0.019	−0.285	0.913	−0.034	0.032	0.116	0.094
JNJ	0.032	−0.004	0.979	0.319	−0.063	0.037	−0.108	0.983	−0.026	0.063	0.145	0.090
JPM	0.018	0.022	0.980	0.435	−0.087	0.057	−0.303	0.944	−0.058	0.064	0.140	0.026
KO	0.031	0.002	0.975	0.385	−0.066	0.030	−0.156	0.957	−0.049	0.062	0.131	0.125
MCD	0.059	0.005	0.984	0.288	−0.038	0.023	−0.300	1.041	−0.058	0.075	0.154	0.076
MMM	0.040	0.014	0.969	0.333	−0.078	0.010	−0.362	1.083	−0.069	0.040	0.146	0.047
MRK	0.027	0.023	0.971	0.334	−0.044	0.015	−0.491	1.087	−0.050	0.051	0.170	0.066
MSFT	0.031	0.025	0.970	0.435	−0.041	0.014	−0.396	1.003	−0.028	0.031	0.128	0.075
PG	0.026	−0.002	0.963	0.373	−0.057	0.027	−0.163	1.103	−0.049	0.058	0.136	0.141
T	0.034	0.012	0.975	0.390	−0.058	0.045	−0.148	0.968	−0.061	0.063	0.164	0.061
UTX	0.045	0.019	0.968	0.350	−0.101	0.037	−0.271	0.962	−0.057	0.067	0.139	0.064
VZ	0.026	0.010	0.979	0.317	−0.060	0.039	−0.191	1.029	−0.054	0.061	0.155	0.027
WMT	0.016	0.005	0.977	0.302	−0.035	0.028	−0.233	1.118	−0.026	0.058	0.133	0.067
XOM	0.040	0.016	0.967	0.343	−0.086	0.041	−0.287	1.092	−0.108	0.048	0.118	0.050
SPY	0.030	−0.004	0.972	0.354	−0.148	0.027	−0.520	1.002	−0.146	0.025	0.122	0.071
Average	0.029	0.018	0.974	0.373	−0.070	0.033	−0.324	1.009	−0.059	0.053	0.140	0.075

Table C.2. Estimates for the realized EGARCH model with GARCH effect in u_t based on the realized kernel (RK) for close-to-close returns. Full sample: January 1, 2002, to December 31, 2013

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	w_u^2	α^u	β^u
AA	−0.008	0.046	0.971	0.328	−0.052	0.042	−0.492	1.040	−0.055	0.065	0.078	0.078	0.365
AIG	−0.026	0.036	0.975	0.486	−0.079	0.039	−0.324	0.875	−0.033	0.048	0.023	0.100	0.789
AXP	0.049	0.014	0.986	0.388	−0.084	0.042	−0.380	0.989	−0.058	0.064	0.044	0.055	0.663
BA	0.069	0.022	0.977	0.300	−0.056	0.036	−0.445	1.052	−0.044	0.078	0.017	0.028	0.856
BAC	0.003	0.027	0.976	0.495	−0.084	0.053	−0.391	0.947	−0.060	0.044	0.017	0.046	0.852
C	−0.016	0.031	0.975	0.457	−0.083	0.068	−0.310	0.919	−0.062	0.067	0.007	0.050	0.907
CAT	0.057	0.033	0.972	0.353	−0.056	0.019	−0.595	1.112	−0.063	0.039	0.051	0.066	0.558
CVX	0.056	0.019	0.967	0.337	−0.075	0.038	−0.338	1.100	−0.101	0.046	0.023	0.068	0.749
DD	0.035	0.020	0.972	0.376	−0.072	0.029	−0.226	0.968	−0.067	0.048	0.062	0.107	0.480
DIS	0.056	0.017	0.981	0.329	−0.071	0.023	−0.341	1.000	−0.069	0.047	0.059	0.090	0.526
GE	0.017	0.012	0.983	0.377	−0.059	0.030	−0.327	0.985	−0.036	0.044	0.021	0.053	0.818
HD	0.044	0.017	0.980	0.377	−0.059	0.030	−0.247	0.956	−0.040	0.051	0.014	0.044	0.858
IBM	0.027	0.011	0.973	0.414	−0.070	0.014	−0.373	0.974	−0.055	0.036	0.013	0.041	0.868
INTC	0.020	0.031	0.975	0.461	−0.049	0.019	−0.282	0.912	−0.034	0.031	0.020	0.058	0.789

(Continued on next page)

Table C.2. Estimates for the realized EGARCH model with GARCH effect in u_t based on the realized kernel (RK) for close-to-close returns. Full sample: January 1, 2002, to December 31, 2013 *(Continued)*

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	w_u^2	α^u	β^u
JNJ	0.033	−0.004	0.979	0.326	−0.060	0.036	−0.111	0.973	−0.025	0.061	0.020	0.046	0.829
JPM	0.020	0.022	0.981	0.434	−0.084	0.056	−0.296	0.937	−0.057	0.063	0.021	0.032	0.823
KO	0.031	0.002	0.973	0.391	−0.065	0.030	−0.156	0.946	−0.047	0.061	0.062	0.106	0.479
MCD	0.060	0.006	0.984	0.295	−0.039	0.023	−0.297	1.038	−0.058	0.074	0.075	0.066	0.487
MMM	0.040	0.014	0.968	0.343	−0.073	0.010	−0.356	1.068	−0.063	0.039	0.023	0.075	0.780
MRK	0.027	0.023	0.971	0.333	−0.039	0.017	−0.495	1.093	−0.048	0.053	0.036	0.071	0.731
MSFT	0.031	0.025	0.970	0.430	−0.037	0.012	−0.393	1.002	−0.026	0.029	0.017	0.063	0.817
PG	0.027	−0.003	0.964	0.367	−0.052	0.025	−0.165	1.087	−0.041	0.057	0.015	0.058	0.847
T	0.035	0.012	0.976	0.385	−0.054	0.044	−0.147	0.966	−0.060	0.066	0.019	0.054	0.840
UTX	0.045	0.019	0.967	0.353	−0.100	0.037	−0.266	0.955	−0.057	0.066	0.034	0.048	0.726
VZ	0.027	0.010	0.980	0.317	−0.059	0.038	−0.189	1.025	−0.051	0.062	0.019	0.024	0.859
WMT	0.015	0.006	0.977	0.303	−0.034	0.028	−0.232	1.114	−0.027	0.058	0.014	0.034	0.867
XOM	0.042	0.016	0.968	0.342	−0.083	0.041	−0.281	1.079	−0.108	0.046	0.012	0.040	0.866
SPY	0.030	−0.003	0.973	0.366	−0.138	0.028	−0.521	0.986	−0.136	0.024	0.006	0.043	0.908
Average	0.030	0.017	0.975	0.374	−0.067	0.032	−0.321	1.004	−0.057	0.052	0.029	0.059	0.748

Table C.3. Improvements in the joint and partial log-likelihood functions, from modeling u_t as an ARCH or GARCH process. The improvement are measured relatively to the standard realized EGARCH(1,1) model that models the variance of u_t to be constant. The results are for the full sample, January 1, 2002, to December 31, 2013, using the realized kernel as the realized measure

	Gains in joint log-likelihood		Gains in partial log-likelihood	
Stocks	ARCH	GARCH	ARCH	GARCH
AA	8.50	11.29	−1.15	−0.93
AIG	34.80	68.70	2.05	4.20
AXP	3.35	10.84	−0.23	−0.50
BA	0.23	9.87	−0.02	−0.22
BAC	20.16	31.39	−3.69	−2.52
C	3.83	47.90	0.60	6.08
CAT	6.08	14.09	−1.08	−1.81
CVX	10.35	25.74	0.66	0.28
DD	16.77	27.12	−1.61	−1.38
DIS	9.79	18.59	−0.16	−0.38
GE	19.06	29.08	−3.34	−2.02
HD	12.32	24.05	0.48	0.46
IBM	4.96	19.80	−0.30	−0.23
INTC	15.37	27.98	−0.29	−0.82
JNJ	11.20	23.12	−0.49	−1.05
JPM	1.05	6.43	0.26	0.52
KO	33.43	37.00	0.37	0.16
MCD	8.27	10.27	−0.08	−0.05
MMM	4.45	35.59	−0.34	0.27
MRK	6.91	27.56	0.30	0.36
MSFT	10.72	26.44	−1.06	−1.00
PG	30.07	45.98	0.56	−0.13
T	5.79	28.05	0.77	1.12
UTX	7.85	13.75	0.05	0.00
VZ	1.44	7.29	0.10	−0.29
WMT	6.56	16.02	−0.03	0.13
XOM	5.34	22.30	−0.39	−0.01
SPY	8.13	33.72	0.20	−0.22
Average	10.96	25.00	−0.28	0.00

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